Applied Analysis (APPM 5450): Midterm 2
8.30am – 9.50am, March 15, 2010. Closed books.

Problem 1: (30 points) Let A be a bounded linear operator on a Hilbert space H.

(a) (10 points) Suppose that $\lambda \in \sigma_{\rm p}(A)$. Prove that $\bar{\lambda} \in \sigma(A^*)$. Can you tell what part of the spectrum $\bar{\lambda}$ belongs to?

(b) (10 points) Suppose that A is self-adjoint, and that M is an invariant subspace of A. Prove that $M^\perp$ is also an invariant subspace of A.

(c) (10 points) Suppose that A is compact and self-adjoint. Which statements are necessarily true?

(i) $\sigma(A) \subseteq \mathbb{R}$.
(ii) $\sigma_{\rm r}(A) = \emptyset$.
(iii) $\sigma_{\rm c}(A) = \emptyset$.
(iv) $\sigma(A) \subseteq \big(\sigma_{\rm p}(A) \cup \{0\}\big)$.
(v) $\sigma(A)$ contains infinitely many points.
(vi) If $\lambda \neq 0$, then $\dim(\ker(A - \lambda\, I)) < \infty$.

No motivation required.

Problem 2: (20 points)

(a) (6 points) Define what is meant by the *derivative* of a distribution $T \in \mathcal{S}^*(\mathbb{R})$.

(b) (14 points) Define $f \in \mathcal{S}^*(\mathbb{R})$ via $f(x) = |x|$. Calculate the distributional derivatives f' and f''. Please motivate carefully.

Problem 3: (20 points) Let $\mathcal{S} = \mathcal{S}(\mathbb{R})$ denote the Schwartz space over $\mathbb{R}$.

(a) (6 points) Define what it means for a sequence to converge in $\mathcal{S}$. If your definition relies on any norms, semi-norms, metrics, bases, *etc*, then state the definition of these.

(b) (8 points) Let α be a positive integer. Prove that $\left(\frac{d}{dx}\right)^\alpha : \mathcal{S} \to \mathcal{S}$ is a continuous map.

(c) (6 points) Set $\varphi_n(x) = e^{-(x-n)^2}$. Does the sequence $(\varphi_n)_{n=1}^\infty$ converge in $\mathcal{S}$? If so, to what?

Problem 4: (30 points) Let H be a Hilbert space with an orthonormal basis $(\varphi_n)_{n=1}^\infty$. Consider the operators

$$A_N\, x = \sum_{n=1}^{N} \frac{1}{n} (\varphi_n,\, x)\, \varphi_n, \qquad \text{and} \qquad B_N\, x = \exp(iA_N) = \sum_{n=1}^{N} e^{i/n} (\varphi_n,\, x)\, \varphi_n.$$

The sequences $(A_N)_{N=1}^\infty$ and $(B_N)_{N=1}^\infty$ have the strong limits A and B, respectively.

(a) (10 points) Put a check-mark in all the boxes that are correct (no motivation required):

	Compact	Self-adjoint	Skew-adjoint	Normal	Unitary	One-to-one	Onto
A_N							
A							
B_N							
B							

(b) (10 points) Do either of the sequences $(A_N)_{N=1}^\infty$ or $(B_N)_{N=1}^\infty$ converge in norm? Motivate your answers.

(c) (10 points) Specify the spectra of A and B and identify their different parts (*i.e.* specify σ_p, $\sigma_{\rm c}$, and $\sigma_{\rm r}$). No motivation required.

Note: *Please staple this page to the front of your exam!*