Applied Analysis (APPM 5450): Midterm 1
8.30am – 9.50am, Feb. 15, 2010. Closed books.

Problem 1: (30p total, 5p per question) Let H denote a Hilbert space with an ON-basis $(e_n)_{n=1}^{\infty}$. Which of the following statements are necessarily true? No motivation required.

(a) $e_n \rightharpoonup 0$.

(b) Suppose that $x,\, x_n \in H$ and $\lim_{n\to\infty} (x_n,\, e_m) = (x,\, e_m)$ for every m. Then $x_n \rightharpoonup x$.

(c) Suppose that $P \in \mathcal{B}(H)$ is such that $P^2 = P$ and $P \neq 0$. Then $||P|| = 1$ if and only if $P^* = P$.

(d) Suppose $A \in \mathcal{B}(H)$ is self-adjoint. Then $C = \exp(iA)$ is unitary.

(e) Suppose that $A, B \in \mathcal{B}(H)$, that A is coercive, and that B is positive. Then $A + B$ is coercive.

(f) Suppose that $A, B \in \mathcal{B}(H)$, and that A is self-adjoint. Then $E = B\,A\,B^*$ is self-adjoint.

Solution (with unrequired motivations):

(a) True.

(b) False. You must also know that $\sup ||x_n|| < \infty$.

(c) True. P is a projection, and for a non-zero projection, we know that:

$$||P|| = 1 \qquad \Leftrightarrow \qquad P = P^* \qquad \Leftrightarrow \qquad \text{ran}(P) = \ker(P)^{\perp}.$$

(d) True.

(e) True. Suppose $x \in H$. Then there is a $c > 0$ such that $(Ax,\, x) \geq c||x||^2$. Then

$$((A+B)x,\, x) = \underbrace{(Ax,\, x)}_{\geq c||x||^2} + \underbrace{(Bx,\, x)}_{>0} \geq c||x||^2.$$

(f) True. $E^* = (B\,A\,B^*)^* = (B^*)^*\,A^*\,B^* = B\,A\,B^* = E$.

Problem 2: (26p) Let $\mathbb{T}$ denote the one-dimensional torus, parameterized with the interval $I = (-\pi,\, \pi]$. Set $e_n(x) = e^{inx}/\sqrt{2\pi}$, and let $\mathcal{P}$ denote the set of all finite linear combinations of basis functions e_n, as usual. Let z denote a non-zero complex number and consider the PDE

$$\frac{\partial u}{\partial t} = z\,\frac{\partial^2 u}{\partial x^2}, \tag{1}$$

along with periodic boundary conditions, and with the initial condition

$$u(x,0) = f(x), \qquad x \in I. \tag{2}$$

(a) (10p) Construct the solution operator $T(t) : \mathcal{P} \to \mathcal{P}$ that maps a function $f \in \mathcal{P}$ to a function $u = T(t)\,f$ that solves (1) and (2).

(b) (8p) Suppose that $t > 0$. For which values of z can the solution operator $T(t)$ be extended to a bounded operator on $L^2(\mathbb{T})$? (Recall that $\mathcal{P}$ is dense in $L^2(\mathbb{T})$.)

(c) (8p) Suppose that $t > 0$ and that z is such that $T(t)$ is a bounded operator on $L^2(\mathbb{T})$. Suppose that $f \in L^2(\mathbb{T})$. For which values of z can you guarantee that $T(t)\,f \in C^1(\mathbb{T})$? Can you ever guarantee that $T(t)\,f \in C^2(\mathbb{T})$?

Solution: Suppose that $f = \sum_{n=-N}^{N} c_n\, e_n$. Then we look for a solution of the form

$$u(x,t) = \sum_{n=-N}^{N} \alpha_n(t)\, e_n(x).$$

Inserting the Ansatz into the PDE (note that it is a finite sum, so differentiating inside the sum is unproblematic), we find (since $\partial_x^2 e_n = -n^2\, e_n$) that

$$\sum_{n=-N}^{N} \alpha_n'(t)\, e_n(x) = \sum_{n=-N}^{N} -z\, n^2\, \alpha_n(t)\, e_n(x).$$

Using that $\alpha_n(0) = c_n$, we find that the solution is

$$\alpha_n(t) = c_n\, e^{-z\, n^2\, t}.$$

(a) Observing that if $f = \sum c_n\, e_n$, then $c_n = (e_n,\, f)$, we see that

$$T(t)f = \sum_{n=-\infty}^{\infty} (e_n\, f)\, e^{-z\, n^2\, t}\, e_n(x).$$

(b) Set $w = \mathrm{Re}(z)$. Then from Parseval, we find

$$||T(t)f||_{L^2}^2 = \sum_{n=-\infty}^{\infty} \left|(e_n\, f)\, e^{-z\, n^2\, t}\right|^2 = \sum_{n=-\infty}^{\infty} e^{-2\, w\, n^2\, t}\, |(e_n\, f)|^2\,.$$

If $w \geq 0$, then $e^{-2\, w\, n^2\, t} \leq 1$, so $||T(t)f|| \leq ||f||$ and $T(t) \in \mathcal{B}(L^2(\mathbb{T}))$. Conversely, if $w < 0$, then

$$||T(t)||_{\mathcal{B}(L^2(\mathbb{T}))} \geq ||T(t)e_n||_{L^2(\mathbb{T})} = e^{-2\, w\, n^2\, t} \to \infty, \qquad \text{as } n \to \infty.$$

Answer: $T(t) \in \mathcal{B}(L^2(\mathbb{T}))$ if and only if $\mathrm{Re}(z) \geq 0$.

(c) By the Sobolev embedding theorem, $H^m(\mathbb{T}) \subseteq C^k(\mathbb{T})$ whenever $m > k + 1/2$. We have

$$||T(t)\, f||_{H^m(\mathbb{T})}^2 = \sum_{n=-\infty}^{\infty} (1 + |n|^{2m})\left|e^{-z\, n^2\, t}\, c_n\right|^2 = \sum_{n=-\infty}^{\infty} (1 + |n|^{2m}) e^{-2\, w\, n^2\, t}\left|c_n\right|^2.$$

Set

$$C = \sup_{n \in \mathbb{Z}} (1 + |n|^{2m})\, e^{-2\, w\, n^2\, t}.$$

If $w > 0$, then $C < \infty$, so

$$||T(t)\, f||_{H^m(\mathbb{T})}^2 \leq C \sum_{n=-\infty}^{\infty} \left|c_n\right|^2 = C\, ||f||_{L^2(\mathbb{T})}^2.$$

We see that $T(t)\, f \in H^m(\mathbb{T})$ for any m, and consequently that $T(t)\, f \in C^k(\mathbb{T})$ for any k.

Answer: $T(t)f \in C^k(\mathbb{T})$ for any $k \geq 0$ whenever $\mathrm{Re}(z) > 0$.

Note: Our analysis is inconclusive for the case $\mathrm{Re}(z) = 0$. As it happens, $T(t)\, f$ is not smooth in this case, but you do not need to show that for a full credit.

Problem 3: (24p) Let H denote a Hilbert space.

(a) (8p) Suppose that $U, T \in \mathcal{B}(H)$, that U is unitary, and that $||T|| = 1/3$. Prove that $A = U + T$ is continuously invertible.

(b) (8p) Suppose that $S \in \mathcal{B}(H)$ and that S is skew-symmetric. Prove that $\text{ran}(I + S)$ is closed.

(c) (8p) For the particular case of $H = L^2(I)$ with $I = [-1, 1]$, give an example of a unitary operator $U \in \mathcal{B}(H)$ and a skew-symmetric operator $S \in \mathcal{B}(H)$ such that $\text{ran}(U + S)$ is not closed.

Solution:

(a) We observe that $A = U\,(I + U^*\,T)$. Now $||U^*Tx|| = ||Tx||$ for any x, so $||U^*T|| = ||T|| = 1/3$. This means that the factor $(I + U^*T)$ is *Neumannable*[1] and

$$(U + T)^{-1} = \big(U\,(I + U^*T)\big)^{-1} = \left(\sum_{n=0}^{\infty}(-U^*T)^n\right) U^*.$$

(b) Let x be any vector. Observe that $(S\,x,\,x) = (x,\,S^*\,x) = -(x,\,S\,x)$. Consequently,

$$||(I + S)\,x||^2 = ||x + S\,x||^2 = ||x||^2 + (S\,x,\,x) + (x,\,S\,x) + ||S\,x||^2 = ||x||^2 + ||S\,x||^2 \geq ||x||^2.$$

Since $I + S$ is *coercive*, it must have closed range.

(c) Define U and S via

$$[U\,f](x) = i\,f(x), \qquad \text{and} \qquad [S\,f](x) = i\,(x - 1)\,f(x).$$

Set $B = U + S$. We have $[B\,f](x) = i\,x\,f(x)$. It remains to prove that B does not have closed range. First observe that the vector $g(x) = 1$ does not belong to $\text{ran}(B)$ (since $1/(i\,x) \notin L^2$). Next observe that for any n, the set $H_n = \{f \in H :\ f(x) = 0 \text{ for } |x| \leq 1/n\}$ does belong to the range, and that $\bigcup_{n=1}^{\infty} H_n$ is dense in H.

Problem 4: (20p) Recall that if A is an $n \times n$ matrix with complex entries, then

$$\text{ran}(A) = \big(\ker(A^*)\big)^{\perp}. \qquad (3)$$

Now suppose that H is a Hilbert space, and $A \in \mathcal{B}(H)$. State and prove a relationship analogous to (3) that A must satisfy.

Solution: Let A be a bounded operator on a Hilbert space H. Then:

$$\begin{aligned}
x \in \text{ran}(A)^{\perp} \quad &\Leftrightarrow \quad (Ay,\,x) = 0 \qquad \forall y \in H,\\
&\Leftrightarrow \quad (y,\,A^*\,x) = 0 \qquad \forall y \in H,\\
&\Leftrightarrow \quad A^*\,x = 0,\\
&\Leftrightarrow \quad x \in \ker(A^*).
\end{aligned}$$

The calculation shows that

$$\ker(A^*) = \text{ran}(A)^{\perp}.$$

Now recall that if V is a linear subspace of H, then $V^{\perp\perp} = \overline{V}$ to obtain

$$\ker(A^*)^{\perp} = \text{ran}(A)^{\perp\perp} = \overline{\text{ran}(A)}.$$

Answer: Let H be a Hilbert space, and let $A \in \mathcal{B}(H)$. Then $\ker(A^*)^{\perp} = \overline{\text{ran}(A)}$.

[1] Recall that if $||B|| < 1$, then $I + B$ is invertible, and $(I + B)^{-1} = \sum_{n=0}^{\infty}(-B)^n$.