Applied Analysis (APPM 5450): Midterm 1

8.30am – 9.50am, Feb. 15, 2010. Closed books.

Problem 1: (30p total, 5p per question) Let H denote a Hilbert space with an ON-basis $(e_n)_{n=1}^{\infty}$. Which of the following statements are necessarily true? No motivation required.

(a) $e_n \rightharpoonup 0$.

(b) Suppose that $x,\, x_n \in H$ and $\lim_{n\to\infty}(x_n,\, e_m) = (x,\, e_m)$ for every m. Then $x_n \rightharpoonup x$.

(c) Suppose that $P \in \mathcal{B}(H)$ is such that $P^2 = P$ and $P \neq 0$. Then $||P|| = 1$ if and only if $P^* = P$.

(d) Suppose $A \in \mathcal{B}(H)$ is self-adjoint. Then $C = \exp(iA)$ is unitary.

(e) Suppose that $A, B \in \mathcal{B}(H)$, that A is coercive, and that B is positive. Then $A + B$ is coercive.

(f) Suppose that $A, B \in \mathcal{B}(H)$, and that A is self-adjoint. Then $E = B\,A\,B^*$ is self-adjoint.

Problem 2: (26p) Let $\mathbb{T}$ denote the one-dimensional torus, parameterized with the interval $I = (-\pi,\, \pi]$. Set $e_n(x) = e^{inx}/\sqrt{2\pi}$, and let $\mathcal{P}$ denote the set of all finite linear combinations of basis functions e_n, as usual. Let z denote a non-zero complex number and consider the PDE

$$\frac{\partial u}{\partial t} = z\,\frac{\partial^2 u}{\partial x^2}, \tag{1}$$

along with periodic boundary conditions, and with the initial condition

$$u(x,0) = f(x), \qquad x \in I. \tag{2}$$

(a) (10p) Construct the solution operator $T(t) : \mathcal{P} \to \mathcal{P}$ that maps a function $f \in \mathcal{P}$ to a function $u = T(t)\,f$ that solves (1) and (2).

(b) (8p) Suppose that $t > 0$. For which values of z can the solution operator $T(t)$ be extended to a bounded operator on $L^2(\mathbb{T})$? (Recall that $\mathcal{P}$ is dense in $L^2(\mathbb{T})$.)

(c) (8p) Suppose that $t > 0$ and that z is such that $T(t)$ is a bounded operator on $L^2(\mathbb{T})$. Suppose that $f \in L^2(\mathbb{T})$. For which values of z can you guarantee that $T(t)\,f \in C^1(\mathbb{T})$? Can you ever guarantee that $T(t)\,f \in C^2(\mathbb{T})$?

Problem 3: (24p) Let H denote a Hilbert space.

(a) (8p) Suppose that $U, T \in \mathcal{B}(H)$, that U is unitary, and that $||T|| = 1/3$. Prove that $A = U + T$ is continuously invertible.

(b) (8p) Suppose that $S \in \mathcal{B}(H)$ and that S is skew-symmetric. Prove that $\text{ran}(I + S)$ is closed.

(c) (8p) For the particular case of $H = L^2(I)$ with $I = [-1,\, 1]$, give an example of a unitary operator $U \in \mathcal{B}(H)$ and a skew-symmetric operator $S \in \mathcal{B}(H)$ such that $\text{ran}(U + S)$ is not closed.

Problem 4: (20p) Recall that if A is an $n \times n$ matrix with complex entries, then

$$\text{ran}(A) = \big(\ker(A^*)\big)^{\perp}. \tag{3}$$

Now suppose that H is a Hilbert space, and $A \in \mathcal{B}(H)$. State and prove a relationship analogous to (3) that A must satisfy.