

Homework set 6 — APPM5450, Spring 2010

If you didn't complete all of the problems 9.1 – 9.11 last week, then continue working on that.

Problem 1: Let H_1 and H_2 be Hilbert spaces, let $U : H_1 \rightarrow H_2$ be unitary, and let $A \in \mathcal{B}(H_1)$. Define $\tilde{A} \in \mathcal{B}(H_2)$ by $\tilde{A} = U A U^{-1}$. Prove that

- $\sigma_p(A) = \sigma_p(\tilde{A})$
- $\sigma_c(A) = \sigma_c(\tilde{A})$
- $\sigma_r(A) = \sigma_r(\tilde{A})$

Problem 2: Let A be a self-adjoint compact operator. For $\lambda \in \rho(A)$, set $R_\lambda = (A - \lambda I)^{-1}$ as usual. Construct the spectral decomposition of R_λ . Use it to prove that

$$\|R_\lambda\| = \frac{1}{\text{dist}(\lambda, \sigma(A))} = \frac{1}{\inf_{\mu \in \sigma(A)} |\lambda - \mu|}.$$

Problem 3: Consider the Hilbert space $H = L^2(I)$, where $I = [-\pi, \pi]$. Define

$$\Omega_t = \{u \in H : u(x) = 0 \forall x \geq t\}.$$

Note that Ω_t is a closed linear subspace of H . Define $P(t)$ as the orthogonal projection onto Ω_t . Consider the self-adjoint operator $A \in \mathcal{B}(H)$ defined by

$$[Au](x) = x u(x).$$

(a) Prove that Ω_t is an invariant subspace of A for every $t \in \mathbb{R}$.

(b) Prove that if $a < b \leq c < d$, then $\text{ran}(P(b) - P(a)) \perp \text{ran}(P(d) - P(c))$. Conclude that for any numbers $-\pi = t_0 < t_1 < t_2 < \dots < t_n = \pi$, it is the case that

$$H = \text{ran}[P(t_1) - P(t_0)] \oplus \text{ran}[P(t_2) - P(t_1)] \oplus \dots \oplus \text{ran}[P(t_n) - P(t_{n-1})],$$

where each term is an invariant subspace of A .

(c) For a positive integer n , set $h = 2\pi/n$, and $\lambda_j = -\pi + h j$. Define the operator

$$(1) \quad A_n = \sum_{j=1}^n \lambda_j (P(\lambda_j) - P(\lambda_{j-1})).$$

Prove that $\|A - A_n\| \leq 2\pi/n$. Conclude that $A_n \rightarrow A$ in norm.

Note: There is a spectral theorem for all self-adjoint operators on Hilbert spaces (all normal ones, even). For operators with continuum spectra such as A , the spectral decomposition of A involves so called “projection valued measures”. The sum (1) is a Riemann-Stieltjes sum of the integral

$$A = \int_{-\pi}^{\pi} \lambda dP(\lambda).$$