Applied Analysis (APPM 5450): Midterm 2
11.35am – 12.50pm, Mar 19, 2008. Closed books.

Problem 1: Let H_1 and H_2 be Hilbert spaces, and let $A \in \mathcal{B}(H_1)$. Suppose further that $U \in \mathcal{B}(H_1, H_2)$ is a unitary map.

(a) Define the following sets: $\rho(A)$, $\sigma(A)$, $\sigma_{\rm p}(A)$, $\sigma_{\rm c}(A)$, $\sigma_{\rm r}(A)$. (4p)

(b) Prove that if $\lambda \in \sigma_{\rm r}(A)$, then $\bar{\lambda} \in \sigma_{\rm p}(A^*)$. (3p)

(c) Define the operator $\hat{A} \in \mathcal{B}(H_2)$ by $\hat{A} = U\,A\,U^{-1}$. Prove that $\sigma_{\rm p}(A) = \sigma_{\rm p}(\hat{A})$. (2p)

(d) Define the operator $\hat{A} \in \mathcal{B}(H_2)$ by $\hat{A} = U\,A\,U^{-1}$. Prove that $\sigma_{\rm c}(A) = \sigma_{\rm c}(\hat{A})$. (2p)

Problem 2: Let $\delta \in \mathcal{S}^*(\mathbb{R})$ denote the Dirac δ-function. Define $T \in \mathcal{S}^*(\mathbb{R})$ via $T(x) = \sin(n\,x)\,\delta'(x)$ where n is an integer, and define $\varphi \in \mathcal{S}(\mathbb{R})$ via $\varphi(x) = (A + B\,x)\,e^{-x^2}$ where A and B are real numbers. Evaluate $\langle \delta', \varphi\rangle$ and $\langle T, \varphi\rangle$. (5p)

Problem 3: Set $H = L^2(I)$ where $I = [-1, 1]$ and let ψ be the function

$$\psi(x) = \begin{cases} -1 & x = -1 \\ 1 + x & x \in (-1, 0) \\ 1 & x \in [0, 1]. \end{cases}$$

Define $A \in \mathcal{B}(H)$ by $[A\,u](x) = \psi(x)\,u(x)$. Draw a graph of ψ. Determine $\sigma(A)$, $\sigma_{\rm p}(A)$, $\sigma_{\rm c}(A)$, and $\sigma_{\rm r}(A)$. No motivation required. (8p)

Problem 4: Let A be a bounded self-adjoint operator on a Hilbert space A. Consider the following statements:

(a) If $\lambda \in \sigma(A)$, then the imaginary part of λ is zero.

(b) The residual spectrum of A is empty.

(c) If M is an invariant subspace of A, then so is $M^\perp$.

(d) The continuous spectrum of A is either empty or consists of the single point 0.

(e) $||A|| = \sup_{||x||=1} |\langle A\,x, x\rangle|$.

(f) If λ and μ are two different eigenvalues of A, then $\ker(A - \lambda\,I) \subseteq \big(\ker(A - \mu\,I)\big)^\perp$.

For each of the six statements, mark whether it is true or false. (2p) for each correct answer.

Extra credit: Pick ***at most two*** *of the statements (4a) – (4f) and either prove them, or give a counterexample. (2p) for each correct proof/counterexample.*

Problem 5: Consider the map $T : \mathcal{S}(\mathbb{R}) \to \mathbb{C}$ defined via $\langle T, \varphi\rangle = \lim_{\varepsilon \searrow 0} \int_{|x|\geq\varepsilon} \frac{1}{x}\,\varphi(x)\,dx$.

(a) Prove that T is continuous. (4p)

(b) Prove that T' is given by $\langle T', \varphi\rangle = \lim_{\varepsilon\searrow 0} \left(-\int_{|x|\geq\varepsilon} \frac{1}{x^2}\,\varphi(x)\,dx + \frac{2\,\varphi(0)}{\varepsilon}\right)$. (4p)