Applied Analysis (APPM 5450): Midterm 1
11.35am – 12.50pm, Feb. 18, 2008. Closed books.

Problem 1: Let H be a Hilbert space with an ON-basis $(\varphi_j)_{j=1}^{\infty}$, and let $(x_n)_{n=1}^{\infty}$, $(y_n)_{n=1}^{\infty}$, $(z_n)_{n=1}^{\infty}$, $(u_n)_{n=1}^{\infty}$, $(v_n)_{n=1}^{\infty}$, and $(w_n)_{n=1}^{\infty}$ be sequences in H for which you know the following:

$\langle x_n, x_m\rangle = 0$ if $m \neq n$ and $\langle x_n, x_n\rangle = 1$.

$||y_n|| = 1$

$\limsup_{n\to\infty} ||z_n|| = \infty$

$||u_n|| = 1/n$ and $\lim_{n\to\infty} \langle \varphi_j, u_n\rangle = 0$ for every j.

$\lim_{n\to\infty} \langle \varphi_j, v_n\rangle = 0$ for every j.

There exists a $w \in H$ such that $||w_n|| \to ||w||$ and $\lim_{n\to\infty} \langle \varphi_j, w_n\rangle = \langle \varphi_j, w\rangle$ for every j.

What can you tell about the convergence properties of these six sequences?

In each of the boxes below, enter a "1", a "2", or a "3", as appropriate.

No motivation is required!

	x_n	y_n	z_n	u_n	v_n	w_n
(1) Necessarily converges strongly. (2) Does not converge strongly. (3) May or may not converge strongly.						
(1) Necessarily has a strongly convergent subsequence. (2) Does not have a strongly convergent subsequence. (3) May or may not have a strongly convergent subsequence.						
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(Note that a "3" indicates that not enough information is provided to determine the convergence property that is asked about.)

2 points for each column that has 4 correct answers and 1 point for each column that has 3 correct answers.

Problem 2: Set $I = [-\pi/2, \pi]$ and consider the Hilbert space $H = L^2(I)$.

(a) Set $\varphi_n(x) = \sin(nx)$ and prove that the set $\mathcal{P} = \text{span}(\varphi_n)_{n=1}^{\infty}$ is not dense in H. (3p)

(b) Set $e_n(x) = e^{i\,n\,x}/\sqrt{2\pi}$ and prove that the set $(e_n)_{n=-\infty}^{\infty}$ is linearly dependent in the sense that there exists a sequence of complex numbers $(\alpha_n)_{n=-\infty}^{\infty}$ such that

$$0 < \sum_{n=-\infty}^{\infty} |\alpha_n|^2 < \infty \qquad \text{and} \qquad \lim_{N\to\infty} ||\sum_{n=-N}^{N} \alpha_n\, e_n||_{L^2(I)} = 0.$$

(4p)

(c) Provide an ON-basis for H. (3p)

Problem 3: Let $(\lambda_n)_{n=-\infty}^{\infty}$ denote a bounded sequence of complex numbers and consider the map

(1) $$A:\ L^2(\mathbb{T}) \to l^2(\mathbb{Z}):\ u \mapsto v = (\dots,\, v_{-1},\, v_0,\, v_1,\, \dots) \text{ where } v_n = \lambda_n\, \langle e_n,\, u\rangle.$$

In (1), e_n denotes the Fourier basis for $L^2(\mathbb{T})$, $e_n(x) = e^{i\,n\,x}/\sqrt{2\,\pi}$.

(a) Prove that $||A|| = \sup_n |\lambda_n|$. (4p)

(b) Let $\mathcal{F}:\ L^2(\mathbb{T}) \to l^2(\mathbb{Z})$ denote the Fourier transform. Complete the following sentences:

$\mathcal{F}^{-1}A$ is *self-adjoint* if and only if every number λ_n satisfies ...

$\mathcal{F}^{-1}A$ is *unitary* if and only if every number λ_n satisfies ...

Motivate briefly. (6p)

Problem 4: Recall that for an $n \times n$ matrix A it is the case that

(2) $$\text{ran}(A) = \ker(A^*)^{\perp}.$$

Now consider the Hilbert space $H = L^2([-\pi,\, \pi])$ and the operator

$$[A\,u](x) = x\,e^{i\,x}\,u(x).$$

(a) Construct A^* and prove that (2) does not hold for A. (6p)

(b) Determine $||A||$. (4p)

Problem 5: Let H_1 and H_2 be Hilbert spaces.

(a) Define what it means for a map $U \in \mathcal{B}(H_1,\, H_2)$ to be *unitary*. (2p)

(b) Suppose that $A \in \mathcal{B}(H_1,\, H_2)$, that A is onto, and that $||A\,u|| = ||u||$ for all $u \in H_1$. Is A necessarily unitary? Motivate briefly. (2p)