

Homework set 8 — APPM5450 Spring 2007 — Solutions/Hints

11.5: Note that

$$\frac{1}{x + i\varepsilon} = \frac{x}{\varepsilon^2 + x^2} - i \frac{\varepsilon}{\varepsilon^2 + x^2}.$$

Fix a $\varphi \in \mathcal{S}$. You need to prove that

$$(1) \quad \lim_{\varepsilon \rightarrow 0} \langle i \frac{\varepsilon}{\varepsilon^2 + x^2}, \varphi \rangle \rightarrow -i\pi\varphi(0).$$

and that

$$(2) \quad \lim_{\varepsilon \rightarrow 0} \langle \frac{x}{\varepsilon^2 + x^2}, \varphi \rangle \rightarrow \langle \text{PV} \left(\frac{1}{x} \right), \varphi \rangle,$$

Proving (1) is simple:

$$\langle i \frac{\varepsilon}{\varepsilon^2 + x^2}, \varphi \rangle = \int_{-\infty}^{\infty} i \frac{\varepsilon}{\varepsilon^2 + x^2} \varphi(x) dx = \{\text{Set } x = \varepsilon y\} = \dots$$

For (2) we need to work a bit more (unless I overlook a simpler solution)

$$\begin{aligned} & \lim_{\varepsilon \rightarrow 0} \langle \frac{x}{\varepsilon^2 + x^2}, \varphi \rangle - \langle \text{PV} \left(\frac{1}{x} \right), \varphi \rangle \\ &= \lim_{\varepsilon \rightarrow 0} \int_{-\infty}^{\infty} \frac{x}{\varepsilon^2 + x^2} \varphi(x) dx - \lim_{\varepsilon \rightarrow 0} \int_{|x| \geq \sqrt{\varepsilon}} \frac{1}{x} \varphi(x) dx \\ &= \lim_{\varepsilon \rightarrow 0} \underbrace{\int_{|x| \geq \sqrt{\varepsilon}} \left(\frac{x}{\varepsilon^2 + x^2} - \frac{1}{x} \right) \varphi(x) dx}_{=S_1} + \lim_{\varepsilon \rightarrow 0} \underbrace{\int_{|x| \leq \sqrt{\varepsilon}} \frac{x}{\varepsilon^2 + x^2} \varphi(x) dx}_{=S_2}. \end{aligned}$$

First we bound $|S_1|$. Note that when $|x| \geq \sqrt{\varepsilon}$, we have

$$\left| \frac{x}{\varepsilon^2 + x^2} - \frac{1}{x} \right| = \frac{\varepsilon^2}{|x|(\varepsilon^2 + x^2)} \leq \frac{\varepsilon^2}{|x|^3} \leq \frac{\varepsilon^2}{\varepsilon^{3/2}} = \sqrt{\varepsilon}.$$

Consequently,

$$\begin{aligned} |S_1| &\leq \limsup_{\varepsilon \rightarrow 0} \int_{|x| \geq \sqrt{\varepsilon}} \left| \frac{x}{\varepsilon^2 + x^2} - \frac{1}{x} \right| |\varphi(x)| dx \\ &\leq \limsup_{\varepsilon \rightarrow 0} \int_{|x| \geq \sqrt{\varepsilon}} \sqrt{\varepsilon} \frac{1}{(1 + |x|^2)} \underbrace{|(1 + |x|^2)\varphi(x)|}_{\leq \|\varphi\|_{0,2}} dx = 0. \end{aligned}$$

In bounding S_2 we use that

$$\int_{|x| \leq \sqrt{\varepsilon}} \frac{x}{\varepsilon^2 + x^2} \varphi(0) dx = 0,$$

and that

$$|\varphi(x) - \varphi(0)| \leq |x| \|\varphi'\|_{\infty} \leq |x| \|\varphi\|_{1,0},$$

to obtain

$$\begin{aligned} |S_2| &= \left| \lim_{\varepsilon \rightarrow 0} \int_{|x| \leq \sqrt{\varepsilon}} \frac{x}{\varepsilon^2 + x^2} (\varphi(x) - \varphi(0)) dx \right| \\ &\leq \limsup_{\varepsilon \rightarrow 0} \int_{|x| \leq \sqrt{\varepsilon}} \underbrace{\frac{|x|}{\varepsilon^2 + x^2} |x|}_{\leq 1} \|\varphi\|_{1,0} dx = 0. \end{aligned}$$

Problem 11.6: We find that

$$\begin{aligned} \langle D(\log|x|) \varphi \rangle &= -\langle \log|x| \varphi' \rangle = -\int_{\mathbb{R}} \log|x| \varphi'(x) dx \\ &= -\lim_{\varepsilon \rightarrow 0} \left\{ \int_{-\infty}^{-\varepsilon} \log(-x) \varphi'(x) dx + \int_{\varepsilon}^{\infty} \log(x) \varphi'(x) dx \right\}. \end{aligned}$$

Partial integrations yield

$$\begin{aligned} -\langle \log|x| \varphi' \rangle &= -\lim_{\varepsilon \rightarrow 0} \left\{ [\log(-x)\varphi(x)]_{-\infty}^{-\varepsilon} - \int_{-\infty}^{-\varepsilon} \frac{1}{-x} \varphi(x) dx + \right. \\ &\quad \left. [\log(x)\varphi(x)]_{\varepsilon}^{\infty} + \int_{\varepsilon}^{\infty} \frac{1}{x} \varphi(x) dx \right\} \\ &= \langle \text{PV}(1/x), \varphi \rangle + \lim_{\varepsilon \rightarrow 0} \{ \log(\varepsilon) (\varphi(\varepsilon) - \varphi(-\varepsilon)) \}. \end{aligned}$$

Since

$$|\varphi(\varepsilon) - \varphi(-\varepsilon)| = \left| \int_{-\varepsilon}^{\varepsilon} \varphi'(x) dx \right| \leq 2\varepsilon \|\varphi\|_{1,0}$$

and $\lim_{\varepsilon \rightarrow 0} \{\varepsilon \log \varepsilon\} = 0$, we find that $\lim_{\varepsilon \rightarrow 0} \{ \log(\varepsilon) (\varphi(\varepsilon) - \varphi(-\varepsilon)) \} = 0$.

Problem 11.7: First prove that $x \cdot \delta(x) = 0$ and that $x \cdot \text{PV}(1/x) = 1$ (using the regular rules for the product between a polynomial and a Schwartz function). Suppose that \cdot is distributive and can pair any two distributions. Then on the one hand we would have

$$\delta(x) \cdot x \cdot \text{PV}(1/x) = \delta(x) \cdot (x \cdot \text{PV}(1/x)) = \delta(x) \cdot 1 = \delta(x).$$

But we would also have

$$\delta(x) \cdot x \cdot \text{PV}(1/x) = (x \cdot \delta(x)) \cdot \text{PV}(1/x) = 0 \cdot \text{PV}(1/x) = 0.$$

This is a contradiction.

Problem 11.8: Fix $\varphi \in \mathcal{S}$. Set $\alpha = \int \varphi$, and define

$$(3) \quad \psi(x) = \int_{-\infty}^x (\varphi(z) - \alpha \omega(z)) dz.$$

Obviously, $\psi \in C^\infty$, and

$$(4) \quad \varphi(x) = \alpha \omega(x) + \psi'(x).$$

Moreover, we find that if $n \geq 1$, then

$$\begin{aligned} \|\psi\|_{n,k} &= \|(1 + |x|^2)^{k/2} \psi^{(n)}\|_{\text{u}} \\ &= \|(1 + |x|^2)^{k/2} (\varphi^{(n-1)} - \alpha \omega^{(n-1)})\|_{\text{u}} \leq \|\varphi\|_{n-1,k} + |\alpha| \|\omega\|_{n-1,k}. \end{aligned}$$

It remains to prove that for any k ,

$$\sup_x (1 + |x|^2)^{k/2} |\psi(x)| < \infty.$$

First consider $x \leq 0$. Then for any k , we have

$$\begin{aligned} &\sup_{x \leq 0} (1 + |x|^2)^{k/2} |\psi(x)| \\ &\leq \limsup_{x \leq 0} \left[(1 + |x|^2)^{k/2} \int_{-\infty}^x \frac{1}{(1 + |y|^{(k+2)/2})} \|\varphi\|_{0,k+2} dy \right. \\ &\quad \left. + |\alpha| (1 + |x|^2)^{k/2} \int_{-\infty}^x \frac{1}{(1 + |y|^{(k+2)/2})} \|\omega\|_{0,k+2} dy \right] < \infty. \end{aligned}$$

To prove the corresponding estimate for $x \geq 0$, we use that since

$$\underbrace{\int_{-\infty}^x (\varphi(z) - \alpha \omega(z)) dz}_{=\psi(x)} + \int_x^{\infty} (\varphi(z) - \alpha \omega(z)) dz = 0,$$

we can also express ψ as

$$\psi(x) = - \int_x^{\infty} (\varphi(z) - \alpha \omega(z)) dz.$$

Then proceed as in the bound for $x \leq 0$.

Problem 1:

$$\begin{aligned} \langle Df, \varphi \rangle &= -\langle f, \varphi' \rangle = - \int_{-\infty}^0 (-x) \varphi'(x) dx - \int_0^{\infty} x \varphi'(x) dx \\ &= \underbrace{[x\varphi(x)]_{-\infty}^0}_{=0} - \int_{-\infty}^0 \varphi(x) dx - \underbrace{[x\varphi(x)]_0^{\infty}}_{=0} + \int_{-\infty}^0 \varphi(x) dx = \langle g, \varphi \rangle, \end{aligned}$$

where

$$g(x) = \begin{cases} -1 & x \leq 0 \\ 1 & x > 0. \end{cases}$$

So $Df = g$. (Note that the value of $g(0)$ is irrelevant, any finite value can be assigned.) Furthermore,

$$\begin{aligned} \langle D^2 f, \varphi \rangle &= \langle Dg, \varphi \rangle = -\langle g, \varphi' \rangle = \int_{-\infty}^0 \varphi'(x) dx - \int_0^{\infty} \varphi'(x) dx \\ &= [\varphi(x)]_{-\infty}^0 - [\varphi(x)]_0^{\infty} = \varphi(0) - (-\varphi(0)) = 2\varphi(0) = \langle 2\delta, \varphi \rangle, \end{aligned}$$

so $D^2 f = 2\delta$.

Problem 2: Assume that f satisfies the given assumptions. We will prove that for any α and k , there exists a number C and a finite integer N such that

$$\|f\varphi\|_{\alpha,k} \leq C \sum_{|\beta|,l \leq N} \|\varphi\|_{\beta,l}.$$

This immediately proves both that $f\varphi \in \mathcal{S}$, and that $f\varphi_n \rightarrow f\varphi$ whenever $\varphi_n \rightarrow \varphi$ in \mathcal{S} .

Fix α and k . Then

$$\begin{aligned} \|f\varphi\|_{\alpha,k} &= \sup_x (1 + |x|^2)^{k/2} |\partial^\alpha (f(x)\varphi(x))| \\ &= \sup_x (1 + |x|^2)^{k/2} \left| \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} (\partial^\gamma f(x)) (\partial^\beta \varphi(x)) \right|. \end{aligned}$$

Now using that for each γ there exist finite numbers N_γ and C_γ such that

$$|\partial^\gamma f(x)| \leq C_\gamma (1 + |x|^2)^{N_\gamma/2}$$

we obtain

$$\begin{aligned} \|f\varphi\|_{\alpha,k} &\leq \sup_x (1 + |x|^2)^{k/2} \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} C_\gamma (1 + |x|^2)^{N_\gamma/2} |(\partial^\beta \varphi(x))| \\ &= \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} C_\gamma \|\varphi\|_{\beta,k+N_\gamma}. \end{aligned}$$

Problem 3: Define for $n = 1, 2, 3, \dots$, the functions

$$\chi_n(x) = \begin{cases} 1 & x \in [n - \frac{1}{4^n}, n], \\ 0 & \text{otherwise,} \end{cases}$$

and set

$$f(x) = \sum_{n=1}^{\infty} 2^n \chi_n(x).$$

Now (2) clearly holds for any k . To prove (3) note that for any given k , we have

$$\begin{aligned} \int_{-\infty}^{\infty} (1 + |x|^2)^{k/2} |f(x)| dx &= \sum_{n=1}^{\infty} \int_{n-4^{-n}}^n (1 + |x|^2)^{k/2} |f(x)| dx \\ &\leq \sum_{n=1}^{\infty} \int_{n-4^{-n}}^n (1 + n^2)^{k/2} 2^n dx = \sum_{n=1}^{\infty} \frac{1}{2^n} (1 + n^2)^{k/2} < \infty. \end{aligned}$$