Weak differentiation on $L^2(\mathbb{T})$ (without Fourier methods)

We consider the space $X = L^2(\mathbb{T})$, with the usual norm

$$||f|| = \left(\int_I |f(x)|^2\,dx\right)^{1/2}.$$

Let Ω denote the set of continuously differentiable functions on $\mathbb{T}$. Note that Ω is dense[1] in X.

Fix a function $f \in X$ and define for $g \in \Omega$ the functional

$$T_f(g) = -\int_{\mathbb{T}} \overline{f(x)}\,g'(x)\,dx.$$

Suppose that there exists a number C (that depends on f) such that

$$|T_f(g)| \leq C\,||g||, \quad \forall\, g \in \Omega.$$

Then T_f is a continuous functional defined on a dense set. It follows that T_f has a unique extension $\tilde{T}_f \in X^*$. By the Riesz representation theorem, we know that there exists a unique $h \in X$ such that

$$\tilde{T}_f(g) = \langle h,\,g\rangle, \quad \forall\, g \in X.$$

We define this function h to be the weak derivative of f.

Remark 1: If f is a classically differentiable function, then our definition of a weak derivative coincides with the classical definition. To see this, note that if $f \in \Omega$, then using integration by parts, we obtain

$$T_f(g) = -\int_{\mathbb{T}} \overline{f(x)}\,g'(x)\,dx = \int_{\mathbb{T}} \overline{f'(x)}\,g(x)\,dx = \langle f',\,g\rangle.$$

It follows that in this case

$$\langle f',\,g\rangle = \langle h,\,g\rangle \quad \forall\, g \in \Omega,$$

and since Ω is dense in X, we must have $f' = h$.

Remark 2: The definition of a weak derivative given here coincides with the Fourier definition. To see this, note that if $g \in \Omega$, and $f = \sum \alpha_n\,e_n$ and $g = \sum \beta_n\,e_n$, then

$$T_f(g) = -\langle f,\,g'\rangle = -\sum_{n\in\mathbb{Z}} \overline{\alpha_n}\,in\beta_n = \sum_{n\in\mathbb{Z}} \overline{in\alpha_n}\,\beta_n.$$

Since Ω is dense in X, it follows that the number

$$C = \sup_{g\in\Omega} |T_f(g)|$$

is finite if and only if $(in\alpha_n)_{n=-\infty}^{\infty} \in L^2(\mathbb{Z})$, and if it is, then necessarily $h = \sum in\alpha_n\,e_n$.

[1] To see that Ω is dense in X, note that Ω contains the set $\mathcal{P}$ of all polynomials on I, and $\mathcal{P}$ is dense in X.