**Applied Analysis (APPM 5450): Midterm 2 – Solutions**

**Problem 1:** Consider the function $f \in \mathcal{S}^*(\mathbb{R})$ defined by

$$f(x) = \begin{cases} -1 & \text{for } x \leq 0, \\ 1 & \text{for } x > 0. \end{cases}$$

Compute the distributional derivative of $f$. (4p)

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We find that

$$\begin{aligned} \langle f',\, \varphi\rangle &= -\langle f,\, \varphi'\rangle = -\int_{-\infty}^{\infty} f(x)\,\varphi'(x)\,dx = \int_{-\infty}^{0} \varphi'(x)\,dx - \int_{0}^{\infty} \varphi'(x)\,dx \\ &= [\varphi(x)]_{-\infty}^{0} - [\varphi(x)]_{0}^{\infty} = \big(\varphi(0) - \lim_{R\to-\infty} \varphi(R)\big) - \big(\lim_{R\to\infty} \varphi(R) - \varphi(0)\big) = 2\varphi(0), \end{aligned}$$

since $\varphi(x) \to 0$ as $|x| \to \infty$. It follows that $f' = 2\delta$ where $\delta$ is the Dirac delta function.

**Problem 2:** Consider the Hilbert space $H = l^2(\mathbb{N})$, and the operators $L, R \in \mathcal{B}(H)$ defined by

$$L\,(x_1,\, x_2,\, x_3,\, \dots) = (x_2,\, x_3,\, x_4,\, \dots),$$
$$R\,(x_1,\, x_2,\, x_3,\, \dots) = (0,\, x_1,\, x_2,\, \dots).$$

In the questions below, $\lambda$ is a complex number,

(a) Prove that if $|\lambda| < 1$, then $\lambda \in \sigma_{\mathrm{p}}(L)$. (2p)

(b) Prove that if $|\lambda| < 1$, then $\lambda \in \sigma_{\mathrm{r}}(R)$. (2p)

(c) Prove that if $|\lambda| = 1$, then $\lambda \in \sigma(L)$. (2p)

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(a) Set $u = (1,\, \lambda,\, \lambda^2, \dots)$. Then $u \in H$, and

$$L\,u = (\lambda,\, \lambda^2,\, \lambda^3,\, \dots) = \lambda\, u,$$

and since $u \neq 0$, it follows that $\lambda \in \sigma_{\mathrm{p}}(L)$.

(b) First we prove that $R - \lambda I$ is one-to-one. Suppose that $(R - \lambda I)\, u = 0$. Then

$$0 = \lambda u_1, \quad u_1 = \lambda u_2, \quad u_2 = \lambda u_3, \quad \text{etc.}$$

If $\lambda = 0$, then $u = 0$. If $\lambda \neq 0$, then the first equation implies that $u_1 = 0$, the second that $u_2 = (1/\lambda)u_1 = 0$, and so on. In either case, $u = 0$, so $\lambda \notin \sigma_{\mathrm{p}}(R)$.

Next note that the range of $R - \lambda I$ is not onto since[1]

$$\mathrm{ran}(R - \lambda I)^{\perp} = \ker(R^* - \bar{\lambda} I) = \ker(L - \bar{\lambda} I) \neq \{0\}$$

since $\bar{\lambda}$ is an eigenvalue of $L$. It follows that $\lambda \notin \rho(R)$. It also follows that the range of $R - \lambda I$ cannot be dense so $\lambda \notin \sigma_{\mathrm{c}}(R)$. We must then have $\lambda \in \sigma_{\mathrm{r}}(R)$.

(c) We proved in (a) that the open unit disc is contained in $\sigma(L)$. Since $\sigma(L)$ is closed, it follows that the closed unit disc must also be contained in $\sigma(L)$.

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[1] In this calculation, we use that $R^* = L$. This is simple to prove:

$$\langle R\,x,\, y\rangle = \sum_{n=2}^{\infty} \overline{x_{n-1}}\, y_n = \sum_{n=1}^{\infty} \overline{x_n}\, y_{n+1} = \langle x,\, L\,y\rangle.$$

**Problem 3:** Let $H$ be a Hilbert space with an ON-basis $(\varphi_n)_{n=1}^{\infty}$. Let $(\lambda_n)_{n=1}^{\infty}$ be a sequence of complex numbers such that $|\lambda_n| < 1$ for every $n$, and let for $N = 1, 2, 3, \dots$ the operator $A_N \in \mathcal{B}(H)$ defined by

$$A_N\, u = \sum_{n=1}^{N} \lambda_n \, \langle \varphi_n,\, u\rangle \, \varphi_n.$$

(a) Prove that there exists an operator $A \in \mathcal{B}(H)$ such that $A_N \to A$ strongly as $N \to \infty$. (2p)

(b) Prove that if some $\lambda_n$ is not purely real, then $A$ is not self-adjoint. (1p)

(c) Specify when, if ever, it is the case that $A_N$ converges to $A$ in norm. (1p)

(d) Suppose that the sequence $(\lambda_n)_{n=1}^{\infty}$ has the cluster point $\lambda$ and that $\lambda \neq 0$. Prove that then $A$ cannot be compact. (2p)

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(a) We need to prove that for any fixed $u \in H$, the sequence the sequence $(A_N\, u)_{N=1}^{\infty}$ has a limit point (in the norm topology) in $H$. Since $H$ is complete, it is sufficient to prove that $(A_N\, u)_{N=1}^{\infty}$ is Cauchy. Suppose that $M$ and $N$ are integers such that $M < N$. We get from Pythagoras that

$$||A_N\, u - A_M\, u||^2 = ||\sum_{n=M+1}^{N} \lambda_n \, \langle \varphi_n,\, u\rangle \, \varphi_n||^2 \leq \sum_{n=M+1}^{N} |\lambda_n|^2 \, |\langle \varphi_n,\, u\rangle|^2.$$

Now use that $|\lambda_n| \leq 1$ for all $n$,

$$||A_N\, u - A_M\, u||^2 \leq \sum_{n=M+1}^{N} |\langle \varphi_n,\, u\rangle|^2 \leq \sum_{n=M+1}^{\infty} |\langle \varphi_n,\, u\rangle|^2. \tag{1}$$

Since $\sum_{n=1}^{\infty} |\langle \varphi_n,\, u\rangle|^2 = ||u||^2 < \infty$, we find that the sum in (1) converges to zero as $M \to \infty$, so $(A_N\, u)$ is indeed Cauchy.

(b) Suppose that $\overline{\lambda_{n_0}} \neq \lambda_{n_0}$. Then

$$\langle A\, \varphi_{n_0},\, \varphi_{n_0}\rangle = \langle \lambda_{n_0}\, \varphi_{n_0},\, \varphi_{n_0}\rangle = \overline{\lambda_{n_0}} \langle \varphi_{n_0},\, \varphi_{n_0}\rangle = \overline{\lambda_{n_0}},$$

but

$$\langle \varphi_{n_0},\, A\, \varphi_{n_0}\rangle = \langle \varphi_{n_0},\, \lambda_{n_0}\, \varphi_{n_0}\rangle = \lambda_{n_0} \langle \varphi_{n_0},\, \varphi_{n_0}\rangle = \lambda_{n_0}.$$

(c) $||A_N - A|| \to 0$ if and only if $\lambda_n \to 0$ as $n \to \infty$.

(d) Pick $\lambda_{n_j}$ such that $\lim\limits_{j \to \infty} \lambda_{n_j} = \lambda \neq 0$, and $|\lambda_{n_j}| \geq |\lambda|/2$ for all $j$. Set

$$u_j = (1/\lambda_{n_j})\, \varphi_{n_j}.$$

Then $||u_j|| \leq 2/|\lambda|$, so $(u_j)$ is a bounded sequence. However, $A\, u_j = \varphi_{n_j}$, so $(A\, u_j)$ is an ON-sequence. It follows that then $A$ cannot be compact. (Recall that if $A$ is compact, and $(u_j)$ is a bounded sequence, then $(A\, u_j)$ must have a norm-convergent subsequence.)

**Problem 4:** Prove that the map $F : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R}) : \varphi \mapsto \varphi^2$ is continuous. (4p)

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Suppose that $\varphi_j \to \varphi$ in $\mathcal{S}$. We need to prove that for each $n, k$, we have

$$\lim_{j\to\infty} ||\varphi_j^2 - \varphi^2||_{n,k} = 0. \tag{2}$$

For $m = 0, 1, 2, 3, \ldots$ set

$$C_m = ||\varphi||_{m,0} + \sup_j ||\varphi_j||_{m,0}.$$

Since $(\varphi_j)$ is Cauchy with respect to $||\cdot||_{m,0}$, each $C_m$ is finite.

Now for a given pair $n$, $k$, we have

$$\begin{aligned}
||\varphi_j^2 - \varphi^2||_{n,k} &= \sup_x \left|(1+x^2)^{k/2}\partial^n(\varphi_j(x)^2 - \varphi(x)^2)\right| \\
&= \sup_x \left|(1+x^2)^{k/2}\partial^n\big((\varphi_j(x) - \varphi(x))(\varphi_j(x) + \varphi(x))\big)\right| \\
&= \sup_x \left|(1+x^2)^{k/2}\sum_{m=0}^{n}\binom{n}{m}\partial^{n-m}(\varphi_j(x) - \varphi(x))\partial^m(\varphi_j(x) + \varphi(x))\right| \\
&\leq \sum_{m=0}^{n}\binom{n}{m}\sup_x\big[(1+x^2)^{k/2}|\partial^{n-m}(\varphi_j(x) - \varphi(x))|\ \underbrace{|\varphi_j^{(m)}(x) + \varphi^{(m)}(x)|}_{\leq C_m}\big] \\
&\leq \sum_{m=0}^{n} C_m \binom{n}{m} ||\varphi_j - \varphi||_{n-m,k}.
\end{aligned}$$

Since the last line involves a finite number of terms, each of which converges to zero as $j \to \infty$, it follows that (2) holds.