Applied Analysis (APPM 5450): Midterm 2
5.00pm – 6.20pm, Mar 19, 2007. Closed books.

Problem 1: Consider the function $f \in \mathcal{S}^*(\mathbb{R})$ defined by

$$f(x) = \begin{cases} -1 & \text{for } x \leq 0, \\ 1 & \text{for } x > 0. \end{cases}$$

Compute the distributional derivative of f. (4p)

Problem 2: Consider the Hilbert space $H = l^2(\mathbb{N})$, and the operators $L, R \in \mathcal{B}(H)$ defined by

$$L\,(x_1,\, x_2,\, x_3,\, \dots) = (x_2,\, x_3,\, x_4,\, \dots),$$
$$R\,(x_1,\, x_2,\, x_3,\, \dots) = (0,\, x_1,\, x_2,\, \dots).$$

In the questions below, λ is a complex number,

(a) Prove that if $|\lambda| < 1$, then $\lambda \in \sigma_{\rm p}(L)$. (2p)

(b) Prove that if $|\lambda| < 1$, then $\lambda \in \sigma_{\rm r}(R)$. (2p)

(c) Prove that if $|\lambda| = 1$, then $\lambda \in \sigma(L)$. (2p)

Problem 3: Let H be a Hilbert space with an ON-basis $(\varphi_n)_{n=1}^\infty$. Let $(\lambda_n)_{n=1}^\infty$ be a sequence of complex numbers such that $|\lambda_n| < 1$ for every n, and let for $N = 1, 2, 3, \dots$ the operator $A_N \in \mathcal{B}(H)$ defined by

$$A_N\, u = \sum_{n=1}^{N} \lambda_n\, \langle \varphi_n,\, u\rangle\, \varphi_n.$$

(a) Prove that there exists an operator $A \in \mathcal{B}(H)$ such that $A_N \to A$ strongly as $N \to \infty$. (2p)

(b) Prove that if some λ_n is not purely real, then A is not self-adjoint. (1p)

(c) Specify when, if ever, it is the case that A_N converges to A in norm. (1p)

(d) Suppose that the sequence $(\lambda_n)_{n=1}^\infty$ has the cluster point λ and that $\lambda \neq 0$. Prove that then A cannot be compact. (2p)

Problem 4: Prove that the map $F :\ \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R}) :\ \varphi \mapsto \varphi^2$ is continuous. (4p)