Applied Analysis (APPM 5450): Midterm 1
5.00pm – 6.20pm, Feb. 19, 2007. Closed books.

Problem 1: Which of the following are true (no motivation required): (2p in total)
(a) In a Hilbert space, any bounded sequence has a weakly convergent subsequence.
(b) If $f, g \in C(\mathbb{T})$, then $||f * g||_{\mathrm{u}} \leq ||f||_{L^2}\,||g||_{L^2}$.
(c) The functions $\big(\sin(nx)\big)_{n=1}^{\infty}$ form an orthogonal basis for $L^2([0,\pi])$.

(a) True - follows from the Banach-Alaoglu theorem.
(b) True - follows from Cauchy-Schwartz ($[f*g](t) = \langle f,\, g_t\rangle$ where $g_t(x) = g(t-x)$).
(c) True - see Exercise 7.3.

Problem 2: Let A be a self-adjoint operator on a Hilbert space H, and let λ be a complex number. Prove that the adjoint of $\lambda\,A$ is $\bar{\lambda}\,A$. For which λ is λA necessarily skew-adjoint? (2p)

For any $x, y \in H$, we find that

$$\langle (\lambda A)\,x,\, y\rangle = \bar{\lambda}\langle A\,x,\, y\rangle = \bar{\lambda}\langle x,\, A^*\,y\rangle = \langle x,\, (\bar{\lambda}A^*)\,y\rangle.$$

Consequently, $(\lambda A)^* = -\lambda A \quad \Leftrightarrow \quad \bar{\lambda} = -\lambda \quad \Leftrightarrow \quad \mathrm{Re}(\lambda) = 0.$

Problem 3: Let u be a function in $L^2(\mathbb{T})$ and set $\alpha_n = \dfrac{1}{\sqrt{2\pi}}\displaystyle\int_{-\pi}^{\pi} e^{-inx}\,u(x)\,dx$, for $n \in \mathbb{Z}$. Obviously, if only finitely many α_n's are non-zero, u will be continuous. Can you give a more general condition involving only the sequence $(\alpha_n)_{n=-\infty}^{\infty}$? (2p)

The Sobolev embedding theorem says that u is continuous if

$$\sum_{n=-\infty}^{\infty} |n|^{2k}\,|\alpha_n|^2 < \infty$$

for some $k > 1/2$.

Problem 4: Let H be a Hilbert space, and let $(\varphi_n)_{n=1}^{\infty}$ be an orthonormal basis for H. Consider for $t \in \mathbb{R}$ the operator $A(t) \in \mathcal{B}(H)$ defined by

$$A(t)\,u = \sum_{n=1}^{\infty} \left(\frac{1+it}{1-it}\right)^n \langle \varphi_n,\, u\rangle\, \varphi_n.$$

(a) Prove that for any $t \in \mathbb{R}$, the operator $A(t)$ is unitary. (2p)

(b) Is it the case that $A(t)$ is either self-adjoint of skew-adjoint for any t? (2p)

(c) For $p \in \mathbb{N}$, set $A_p = A(1/p)$. Does the sequence $(A_p)_{p=1}^{\infty}$ converge in $\mathcal{B}(H)$? If so, specify in which sense, and what the limit is. Motivate your answer. (4p)

Set $\lambda_n(t) = \left(\dfrac{1+it}{1-it}\right)^n$.

It follows immediately from Parseval's equality that

$$(1) \qquad A(t)^*\,u = \sum_{n=1}^{\infty} \overline{\lambda_n(t)}\,\langle \varphi_n,\, u\rangle\,\varphi_n = \sum_{n=1}^{\infty} \lambda_n(-t)\,\langle \varphi_n,\, u\rangle\,\varphi_n = A(-t)\,u.$$

(a) Since $\lambda_n(t)^{-1} = \lambda_n(-t)$, it follows that $A(t)$ is invertible and that $A(t)^{-1} = A(-t)$. That $A(t)$ is unitary is now obvious since $A(t)^* = A(-t) = A(t)^{-1}$.

(b) We find that $A(t)$ is self-adjoint iff every $\lambda_n(t)$ is a real number. This happens only for $t = 0$. Similarly, $A(t)$ is skew-adjoin iff every $\lambda_n(t)$ is a purely imaginary number. That never happens.

(c) A_p converges strongly to the identity operator, but it does not converge in norm.

We first prove that $A_p \to I$ strongly. Fix $u \in H$. Fix $\varepsilon > 0$. Pick an N such that $\sum_{n>N} |\langle \varphi_n,\, u\rangle|^2 < \varepsilon$. Then, using Parseval we find that

$$\begin{aligned}
&\limsup_{p\to\infty} ||A(1/p)u - u||^2 \\
&\quad = \limsup_{p\to\infty} \left(\sum_{n=1}^{N} \big|\lambda_n(1/p) - 1\big|^2 |\langle \varphi_n,\, u\rangle|^2 + \sum_{n=N+1}^{\infty} \underbrace{\big|\lambda_n(1/p) - 1\big|^2}_{\leq 2} |\langle \varphi_n,\, u\rangle|^2 \right) \\
&\qquad\qquad \leq \sum_{n=1}^{N} \underbrace{\left(\limsup_{p\to\infty} \big|\lambda_n(1/p) - 1\big|^2\right)}_{=0} |\langle \varphi_n,\, u\rangle|^2 + 2 \underbrace{\sum_{n=N+1}^{\infty} |\langle \varphi_n,\, u\rangle|^2}_{<\varepsilon} < 2\varepsilon.
\end{aligned}$$

Since ε was arbitrary, it follows that $\lim_{p\to\infty} ||A_p u - u|| = 0$.

To prove that A_p cannot converge in norm to I, simply pick for any $p > 0$, an $n \in \mathbb{N}$ such that $|\lambda_n(1/p) - 1| \geq 1/2$. Then

$$||A_p - I|| = \sup_{||u||=1} ||A_p u - u|| \geq ||A_p \varphi_n - \varphi_n|| = ||(\lambda_n(1/p) - 1)\,\varphi_n|| \geq 1/2.$$

Problem 5: Consider the Hilbert space $H = L^2(\mathbb{T})$, and the operator $A \in \mathcal{B}(H)$ defined by $[A\,u](x) = (1 + \cos x)\,u(x)$. Prove that A is self-adjoint and positive, but not coercive. (5p)

Set $\varphi(x) = 1 + \cos(x)$.

That A is self-adjoint follows immediately from the fact that $1 + \cos x$ is real:

$$\langle A\,u,\, v\rangle = \int_{-\pi}^{\pi} \overline{(1 + \cos x)\,u(x)}\,v(x)\,dx = \int_{-\pi}^{\pi} \overline{u(x)}\,\big((1 + \cos x)\,v(x)\big)\,dx = \langle u,\, A\,v\rangle.$$

That A is non-negative follows from the fact that $1 + \cos x$ is non-negative:

$$(2) \qquad \langle A\,u,\, u\rangle = \int_{-\pi}^{\pi} (1 + \cos x)|u(x)|^2\,dx \geq 0.$$

To further prove that A is positive, note that if we have equality in (2), then $u(x)$ must be zero everywhere except possibly on a set of measure zero, since $1 + \cos x$ is zero only for $x = \pm\pi$.

Recall that A is coercive iff

$$\inf_{||u||=1} \langle A\,u,\, u\rangle > 0.$$

To prove that this is not true, define the functions $u_n \in H$ by

$$u_n(x) = \begin{cases} \sqrt{n} & x \in [\pi - 1/n,\, \pi], \\ 0 & x \in (-\pi,\, \pi - 1/n). \end{cases}$$

Note that $||u_n|| = 1$, so

$$\begin{aligned} \inf_{||u||=1} \langle A\,u,\, u\rangle \leq \inf_{n\in\mathbb{N}} \langle A\,u_n,\, u_n\rangle &= \inf_{n\in\mathbb{N}} \int_{\pi-1/n}^{\pi} (1 + \cos x)\,|u_n(x)|^2\,dx \\ &\leq \inf_{n\in\mathbb{N}} \int_{\pi-1/n}^{\pi} \big(1 + \cos(\pi - 1/n)\big)\,n\,dx = \inf_{n\in\mathbb{N}} \big(1 + \cos(\pi - 1/n)\big) = 0. \end{aligned}$$

Problem 6: Consider the Hilbert space $H = L^2(\mathbb{R})$. For this problem, we define H as the closure of the set of all compactly supported smooth functions on $\mathbb{R}$ under the norm

$$||u|| = \Big(\int_{-\infty}^{\infty} |u(x)|^2\, dx\Big)^{1/2}.$$

Which of the following sequences converge weakly in H? Motive your answers briefly. (2p each)

(a) $(u_n)_{n=1}^{\infty}$ where $u_n(x) = \begin{cases} |x-n|, & \text{for } x \in [n-1,\, n+1], \\ 0, & \text{for } x \in (-\infty, n-1) \cup (n+1, \infty). \end{cases}$

(b) $(v_n)_{n=1}^{\infty}$ where $v_n(x) = \sin(n\,x)\, e^{-x^2}$.

(c) $(w_n)_{n=1}^{\infty}$ where $w_n(x) = e^{-x^2/n}$.

Remark: Note that there exist functions f and f_n in H such that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x)\, f_n(x)\, dx \neq \int_{-\infty}^{\infty} f(x)\, \big(\lim_{n\to\infty} f_n(x)\big)\, dx.$$

Keeping in mind the definition of H given above, you can solve the problem without having to make such interchanges (not using any Lebesgue integrals at all).

Recall that if a sequence $(\varphi_n)_{n=1}^{\infty}$ is bounded, and there exists a function $\varphi \in H$ such that $\langle \varphi_n, \psi\rangle \to \langle \varphi, \psi\rangle$ for all ψ in a dense subset $\mathcal{P}$, then $\varphi_n \rightharpoonup \varphi$. In (a) and (b), we let $\mathcal{P}$ be the set of compactly supported smooth functions (this is dense *by definition*).

(a) Since $u_n(x) = u_1(x-n+1)$, it follows that $||u_n|| = ||u_1||$ and so (u_n) is a bounded sequence. Furthermore, if $\psi \in \mathcal{P}$, then $\langle u_n, \psi\rangle \to 0$ since for large enough n, the support of u_n will be outside the support of ψ. It follows that $u_n \rightharpoonup 0$.

(b) $||v_n||^2 = \int_{-\infty}^{\infty} |\sin(nx)|^2\, e^{-2\,x^2}\, dx \leq \int_{-\infty}^{\infty} e^{-2\,x^2}\, dx$ so (v_n) is bounded. Furthermore, if $\psi \in \mathcal{P}$, then

$$\begin{aligned} |\langle v_n, \psi\rangle| &= \left|\int_{-\infty}^{\infty} \sin(nx)\, e^{-x^2}\, \psi(x)\, dx\right| = \{\text{partial integration}\} \\ &= \left|\int_{-\infty}^{\infty} \frac{1}{n}\cos(nx)\, \frac{d}{dx}\Big(e^{-x^2}\,\psi(x)\Big)\, dx\right| \leq \frac{1}{n}\int_{-\infty}^{\infty} \left|\frac{d}{dx}\Big(e^{-x^2}\,\psi(x)\Big)\right|\, dx \to 0, \end{aligned}$$

so $v_n \rightharpoonup 0$ (the boundary terms vanish since ψ has compact support).

(c) $||w_n||^2 = \int_{-\infty}^{\infty} e^{-2\,x^2/n}\, dx = \{x = \sqrt{n}\, y\} = \sqrt{n}\int_{-\infty}^{\infty} e^{-2\,x^2}\, dx = \sqrt{n}||w_1||^2 \to \infty$ so (w_n) cannot converge weakly.