Applied Analysis (APPM 5450): Midterm 1
5.00pm – 6.20pm, Feb. 19, 2007. Closed books.

Problem 1: Which of the following are true (no motivation required): (2p in total)
(a) In a Hilbert space, any bounded sequence has a weakly convergent subsequence.
(b) If $f, g \in C(\mathbb{T})$, then $||f * g||_{\mathrm{u}} \leq ||f||_{L^2}\, ||g||_{L^2}$.
(c) The functions $\big(\sin(nx)\big)_{n=1}^{\infty}$ form an orthogonal basis for $L^2([0,\pi])$.

Problem 2: Let A be a self-adjoint operator on a Hilbert space H, and let λ be a complex number. Prove that the adjoint of $\lambda\, A$ is $\bar{\lambda}\, A$. For which λ is λA necessarily skew-adjoint? (2p)

Problem 3: Let u be a function in $L^2(\mathbb{T})$ and set $\alpha_n = \dfrac{1}{\sqrt{2\pi}} \displaystyle\int_{-\pi}^{\pi} e^{-inx}\, u(x)\, dx$, for $n \in \mathbb{Z}$. Obviously, if only finitely many α_n's are non-zero, u will be continuous. Can you give a more general condition involving only the sequence $(\alpha_n)_{n=-\infty}^{\infty}$? (2p)

Problem 4: Let H be a Hilbert space, and let $(\varphi_n)_{n=1}^{\infty}$ be an orthonormal basis for H. Consider for $t \in \mathbb{R}$ the operator $A(t) \in \mathcal{B}(H)$ defined by

$$A(t)\, u = \sum_{n=1}^{\infty} \left(\frac{1+it}{1-it}\right)^n \langle \varphi_n,\, u\rangle\, \varphi_n.$$

(a) Prove that for any $t \in \mathbb{R}$, the operator $A(t)$ is unitary. (2p)

(b) Is it the case that $A(t)$ is either self-adjoint of skew-adjoint for any t? (2p)

(c) For $p \in \mathbb{N}$, set $A_p = A(1/p)$. Does the sequence $(A_p)_{p=1}^{\infty}$ converge in $\mathcal{B}(H)$? If so, specify in which sense, and what the limit is. Motivate your answer. (4p)

Problem 5: Consider the Hilbert space $H = L^2(\mathbb{T})$, and the operator $A \in \mathcal{B}(H)$ defined by $[A\, u](x) = (1 + \cos x)\, u(x)$. Prove that A is self-adjoint and positive, but not coercive. (5p)

Problem 6: Consider the Hilbert space $H = L^2(\mathbb{R})$. For this problem, we define H as the closure of the set of all compactly supported smooth functions on $\mathbb{R}$ under the norm

$$||u|| = \left(\int_{-\infty}^{\infty} |u(x)|^2\, dx\right)^{1/2}.$$

Which of the following sequences converge weakly in H? Motive your answers briefly. (2p each)

(a) $(u_n)_{n=1}^{\infty}$ where $u_n(x) = \begin{cases} |x-n|, & \text{for } x \in [n-1,\, n+1], \\ 0, & \text{for } x \in (-\infty, n-1) \cup (n+1, \infty). \end{cases}$

(b) $(v_n)_{n=1}^{\infty}$ where $v_n(x) = \sin(n\, x)\, e^{-x^2}$.

(c) $(w_n)_{n=1}^{\infty}$ where $w_n(x) = e^{-x^2/n}$.

Remark: Note that there exist functions f and f_n in H such that

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x)\, f_n(x)\, dx \neq \int_{-\infty}^{\infty} f(x)\, \big(\lim_{n\to\infty} f_n(x)\big)\, dx.$$

Keeping in mind the definition of H given above, you can solve the problem without having to make such interchanges (not using any Lebesgue integrals at all).