Homework set 13 — APPM5450, Spring 2007

From the textbook: 12.4. Problems 1 and 2 below are important, problem 3 less so.

Problem 1: Let $(f_n)_{n=1}^{\infty}$ be a sequence of real valued measurable functions on $\mathbb{R}$ such that $\lim_{n\to\infty} f_n(x) = x$ for all $x \in \mathbb{R}$. Specify which of the following limits necessarily exist, and give a formula for the limit in the cases where this is possible:

$$\lim_{n\to\infty} \int_1^2 \frac{f_n(x)}{1 + f_n(x)^2}\, dx, \tag{1}$$

$$\lim_{n\to\infty} \int_0^1 \frac{\sin(f_n(x))}{f_n(x)}\, dx, \tag{2}$$

$$\lim_{n\to\infty} \int_0^\infty \frac{\sin(f_n(x))}{f_n(x)}\, dx, \tag{3}$$

$$\lim_{N\to\infty} \int_0^1 \sum_{n=1}^{N} \frac{|f_n(x)|}{n^2(1 + |f_n(x)|)}\, dx, \tag{4}$$

$$\lim_{N\to\infty} \int_0^\infty \sum_{n=1}^{N} \frac{1}{n^2(1 + |f_n(x)|^2)}\, dx. \tag{5}$$

Problem 2: Let $(f_n)_{n=1}^{\infty}$ be a sequence of real-valued measurable functions on $\mathbb{R}$ such that $|f_n(x)| \leq 1$ and $\lim_{n\to\infty} f_n(x) = 1$ for all x. Evaluate

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n(\cos x)\, e^{-\frac{1}{2}(x-2\pi n)^2}\, dx.$$

Make sure to justify your calculation.

Problem 3: Let (X, μ) be a measure space and consider the space $L^\infty(X, \mu)$ consisting of all measurable functions from X to $\mathbb{R}$ such that

$$||f||_\infty = \operatorname*{ess\,sup}_{x\in X} |f(x)| < \infty.$$

Prove that $L^\infty(X, \mu)$ is closed under the norm $||\cdot||_\infty$.

Hint: You may want to start as follows:

(1) Let $(f_n)_{n=1}^{\infty}$ be a Cauchy sequence in $L^\infty(X, \mu)$.
(2) For each positive integer k, there exists and N_k such that for $m, n \geq N_k$, $||f_n - f_m||_\infty < 1/k$.
(3) For each k, and for each $m, n \geq N_k$, let Ω_{mn}^k denote the set of all $x \in X$ such that $|f_m(x) - f_n(x)| < 1/k$. What can you tell about Ω_{mn}^k in light of (2)?
(4) Form $\Omega^k = \cap_{m,n=N_k}^{\infty} \Omega_{mn}^k$. What do you know about Ω^k in view of your conclusion from (3)?
(5) Form $\Omega = \cap_{k=1}^{\infty} \Omega^k$. What do you know about Ω in view of your conclusion from (4)?
(6) What can you tell about $(f_n(x))_{n=1}^{\infty}$ for $x \in \Omega$?