Homework set 12 — APPM5450, Spring 2007 — Hints

Problem 11.22: Set $T = \text{sign}(t)$. We seek to prove that $\check{T} = \alpha \text{PV}(1/x)$ for some constant α.

For $N = 1, 2, 3, \dots$, set $T_N = \chi_{[-N,N]}\, T$. Then $T_N \to T$ in $\mathcal{S}'$ (prove this!). Since the Fourier transform is a continuous operator on $\mathcal{S}'$, we then know that $\check{T}$ is the limit of the sequence $(\check{T}_N)_{N=1}^{\infty}$.

Since $T_N \in L^1$, we can compute $\check{T}_N$ by directly evaluating the integral. We find that

$$\check{T}_N(x) = \beta \frac{1 - \cos(N\,x)}{x}$$

for some constant β. In a distributional sense, this is equivalent to saying that

$$\check{T}_N(x) = \beta\,\text{PV}(1/x) - \beta\,\cos(N\,x)\,\text{PV}(1/x).$$

It only remains to prove that $\cos(N\,x)\,\text{PV}(1/x) \to 0$ in $\mathcal{S}'$. We find that

$$\lim_{N\to\infty} \langle \cos(N\,x)\,\text{PV}(1/x),\, \varphi\rangle = \lim_{N\to\infty} \langle \text{PV}(1/x),\, \cos(N\,x)\,\varphi\rangle = \cdots =$$
$$\lim_{N\to\infty} \int_0^\infty \cos(N\,x)\,\frac{\varphi(x) - \varphi(-x)}{x}\,dx = \cdots \text{partial integration} \cdots = 0.$$

Lots of details to fill in ...

Problem 12.2:

(a) Use that $A\backslash B = A \cap B^{\text{c}} = (A^{\text{c}} \cup B)^{\text{c}}$.

(b) Split B into two well-chosen disjoint sets and use additivity.

(c) Split $A \cup B$ into three well-chosen disjoint sets and use additivity. (I think we did this one in class.)

Problem 12.3: The trick is to write $\bigcup_{n=1}^{\infty} A_n$ as a disjoint union. For $n = 1, 2, 3, \dots$ set $B_n = A_{n+1}\backslash A_n$. Then

$$\bigcup_{n=1}^{\infty} A_n = A_1 \cup \left(\bigcup_{n=1}^{\infty} B_n\right),$$

where there union on the right is a disjoint one. Now use additivity twice:

$$\mu\left(\bigcup_{n=1}^{\infty} A_n\right) = \mu\left(A_1 \cup \left(\bigcup_{n=1}^{\infty} B_n\right)\right) = \mu(A_1) + \sum_{n=1}^{\infty} \mu(B_n)$$
$$= \lim_{N\to\infty} \left(\mu(A_1) + \sum_{n=1}^{N} \mu(B_n)\right) = \lim_{N\to\infty} \mu\left(A_1 \cup \left(\bigcup_{n=1}^{N} B_n\right)\right) = \lim_{N\to\infty} \mu\left(A_N\right).$$

For the second part, set $C = \cap_{n=1}^{\infty} A_n$ and $C_n = A_n\backslash A_{n+1}$. Then

$$\mu(A_N) = \mu\left(C \cup \left(\bigcup_{n=N}^{\infty} C_n\right)\right) = \mu(C) + \sum_{n=N}^{\infty} \mu(C_n).$$

Since $\infty > \mu(A_1) \geq \sum_{n=1}^{\infty} \mu(C_n)$, we find that

$$\lim_{N \to \infty} \sum_{n=N}^{\infty} \mu(C_n) = 0,$$

which completes the proof. For the counterexample, consider $X = \mathbb{R}^2$, and $A_n = \{x = (x_1, x_2) : |x_2| < 1/n\}$. Then $\mu(A_n) = \infty$ for all n, but $\cap_{n=1}^{\infty} A_n$ is the x_1-axis, which has measure zero.

Problem 12.5: Straight-forward.

Problem 12.7:

Reflexivity: It is obvious that $f(x) = f(x)$ a.e.

Symmetry: If $f(x) = g(x)$ a.e., then obviously $g(x) = f(x)$ a.e.

Transitivity: Suppose that $f(x) = g(x)$ a.e. and that $g(x) = h(x)$ a.e. Set

$$\begin{aligned} A &= \{x : f(x) \neq g(x)\} \\ B &= \{x : g(x) \neq h(x)\} \\ C &= \{x : f(x) \neq h(x)\}. \end{aligned}$$

We know that $\mu(A) = \mu(B) = 0$, and we want to prove that $\mu(C) = 0$. It is clearly the case that $C \subseteq A \cup B$, and then it follows directly that $\mu(C) \leq \mu(A) + \mu(B) = 0$.