Applied Analysis (APPM 5450): Final
4.30pm – 7.00pm, May 7, 2007. Closed books.

Problem 1: No motivation required for these questions. 2p each.

(a) State Hölder's inequality.

(b) Define what it means for a sequence $(\varphi_n)_{n=1}^{\infty}$ of Schwartz functions to converge in $\mathcal{S}(\mathbb{R})$.

(c) Let H be a Hilbert space, and let $(A_n)_{n=1}^{\infty}$ be a sequence of operators in $\mathcal{B}(H)$. Define what it means for A_n to converge *strongly* to some operator $A \in \mathcal{B}(H)$.

(d) Let (X, μ) be a σ-finite measure space. For which numbers p in the interval $[1, \infty]$ is it necessarily the case that $(L^p(X, \mu))^* = L^q(X, \mu)$, where q is such that $(1/p) + (1/q) = 1$. For which numbers p is $L^p(X, \mu)$ necessarily reflexive?

(e) Let H be a Hilbert space, and let A be a linear bounded operator on H. Give a formula that relates the range of A to the kernel of A^*.

(f) Let H be a Hilbert space and let $A \in \mathcal{B}(H)$ be a self-adjoint operator. Let H_1 be an invariant subspace of A. Is $H_1^{\perp}$ necessarily an invariant subspace of A? Is $H_1^{\perp}$ necessarily an invariant subspace of A if A is skew-adjoint instead of self-adjoint?

(g) Let H be a Hilbert space, and let $A \in \mathcal{B}(H)$ be self-adjoint and compact. What can you say about $\sigma_{\rm c}(A)$?

Problem 2: Let H be a Hilbert space, and let $P \in \mathcal{B}(H)$ be an operator such that $P^2 = P$. Prove that the statements (S1) and (S2) given below are equivalent: (4p)

(S1): $\quad (\mathrm{ran}(P))^{\perp} = \ker(P)$.

(S2): $\quad \langle P\,x,\, y\rangle = \langle x,\, P\,y\rangle$ for all $x, y \in H$.

Problem 3: Let $\delta \in \mathcal{S}(\mathbb{R})^*$ denote the Dirac delta-function as usual, let δ' denote the distributional derivative of δ, and define for a positive integer n the distribution $T_n \in \mathcal{S}(\mathbb{R})^*$ by $T_n(x) = \sin(n\,x)\,\delta'(x)$.

(a) Calculate the Fourier transform $\hat{T}_n$ of T_n. (2p)

(b) Does the sequence $(\hat{T}_n)_{n=1}^{\infty}$ converge in $\mathcal{S}(\mathbb{R})^*$? (2p)

Hint: You may want to start by simplifying the expression for T_n.

Problem 4: Let $p \in [1, \infty)$, let g be a function in $L^p(\mathbb{R})$, and let $(f_n)_{n=1}^{\infty}$ be measurable functions from $\mathbb{R}$ to $\mathbb{R}$ such that

$$\sum_{n=1}^{\infty} |f_n(x)| \leq g(x), \qquad \text{a.e.}$$

Set $h_N = \sum_{n=1}^{N} f_n$. Prove that the sequence $(h_N)_{N=1}^{\infty}$ converges in $L^p(\mathbb{R})$. (5p)

Problem 5: Consider the Hilbert space $H = L^2(\mathbb{T})$, let $a \in (0, \pi)$ be a real number, and define the operator $T \in \mathcal{B}(H)$ by

$$[T\,u](x) = \frac{1}{2}\big(u(x-a) + u(a-x)\big).$$

(a) Construct T^* and indicate whether T is self-adjoint. (2p)

(b) Prove that T is not unitary. Is T normal? (2p)

(c) Specify infinite dimensional subspaces H_1 and H_2 of H such that the map $T:\ H_1 \to H_2$ is a unitary operator. (2p)

(d) Let $\mathcal{F}:\ H \to l^2(\mathbb{Z})$ denote the Fourier transform. Determine the operator $\hat{T} : l^2(\mathbb{Z}) \to l^2(\mathbb{Z})$ given by $\hat{T} = \mathcal{F}\,T\,\mathcal{F}^{-1}$. (2p)

(e) Determine $\sigma(T)$. As far as you can, classify the different parts of the spectrum as belonging to the point, continuous, or residual spectrum. (3p)

Hint: You may want to attempt question 5(e) last as it could be time-consuming.