Applied Analysis (APPM 5450): Midterm 2

5.00pm – 6.20pm, Mar 22, 2006. Closed books.

Note: The problems are worth two points each, for a total of 16 points.

Problem 1: In this problem, $\partial = (d/dx)$, and $\delta \in \mathcal{S}^*(\mathbb{R})$ denotes the Dirac delta function.

(a) For $T \in \mathcal{S}^*(\mathbb{R})$, define ∂T, and prove that what you define is a continuous functional on $\mathcal{S}(\mathbb{R})$. (You may use the fact that $\partial : \mathcal{S} \to \mathcal{S}$ is continuous.)

(b) Set $U(x) = x\,[\partial\delta](x)$, and calculate, for $\varphi \in \mathcal{S}$, $\langle U, \varphi\rangle$.

(c) Set $V(x) = x\,\delta(x)$, and calculate, for $\varphi \in \mathcal{S}$, $\langle \partial V, \varphi\rangle$.

Solution:

(a) For $T \in \mathcal{S}^*$, we define ∂T by $\langle \partial T,\, \varphi\rangle = -\langle T,\, \partial\varphi\rangle$.

To prove that ∂T is a continuous functional, we need to prove that when $\varphi_n \to \varphi$ in $\mathcal{S}$, $\langle \partial T,\, \varphi_n\rangle \to \langle \partial T,\, \varphi\rangle$ in $\mathbb{R}$. To do this, we assume that $\varphi_n \to \varphi$ in $\mathcal{S}$. Then

$$\langle \partial T,\, \varphi_n\rangle = -\langle T,\, \partial\varphi_n\rangle \to -\langle T,\, \partial\varphi\rangle = \langle \partial T,\, \varphi_n\rangle.$$

The first and the last steps are simply the definition of ∂T. The middle step is valid since T is continuous, and $\partial\varphi_n \to \partial\varphi$ in $\mathcal{S}$. (Linearity is obvious.)

(b) We have

$$\langle x\,\partial\delta,\, \varphi\rangle = \langle \partial\delta,\, x\varphi\rangle = -\langle \delta,\, \partial(x\varphi)\rangle = -\langle \delta,\, \varphi + x\varphi'\rangle = -\varphi(0) - 0\varphi'(0) = -\varphi(0).$$

(c) We have

$$\langle \partial(x\delta),\, \varphi\rangle = -\langle x\delta,\, \varphi'\rangle = -\langle \delta,\, x\varphi'\rangle = -0\,\varphi'(0) = 0.$$

Note 1: You could alternatively have shown that $V = 0$ since $\langle V,\, \varphi\rangle = \langle x\delta,\, \varphi\rangle = \langle \delta,\, x\varphi\rangle = 0$; then trivially $\partial V = \partial 0 = 0$.

Note 2: There is no product rule for differentiating products of distributions. (In fact, there is no general product of distributions...)

Problem 2: We define the functions $\varphi_n \in \mathcal{S}$ by setting $\varphi_n(x) = \frac{x^2}{\sqrt{x^2+1/n}} e^{-x^2}$. Does the sequence converge in $\mathcal{S}$ as $n \to \infty$? If so, to what?

Solution: The sequence φ_n converges in the uniform norm to $\varphi(x) = |x|\, e^{-x^2}$. Since φ is not a Schwartz function, the sequence φ_n cannot converge in $\mathcal{S}$.

(To prove the last assertion, pick any $\psi \in \mathcal{S}$. Then $\lim ||\varphi_n - \psi||_{0,0} = \lim ||\varphi_n - \psi||_{\mathrm{u}} = ||\varphi - \psi||_{\mathrm{u}} > 0$, so clearly φ_n cannot converge to ψ.)

Problem 3: Let H be a Hilbert space and let A be a compact self-adjoint operator on H. Let b be a non-zero real number, and set $f(x) = (x - i\,b)^{-1}$ where i is the imaginary unit. This question concerns different ways of defining $f(A)$.

(a) Noting that f has the MacLaurin expansion $f(x) = (-1/ib)\sum_{n=0}^{\infty}(x/ib)^n$, we define $B_N = (-1/ib)\sum_{n=1}^{N}((1/ib)\,A)^n$. Describe when, if ever, the sequence $(B_N)_{N=1}^{\infty}$ converges in norm in $\mathcal{B}(H)$.

(b) Let $(\varphi_n)_{n=1}^{\infty}$ denote an orthonormal basis for H consisting of eigenvectors of A, so that $A\,\varphi_n = \lambda_n\,\varphi_n$. Define the operator C_N by setting, for $u \in H$, $C_N\,u = \sum_{n=1}^{N} f(\lambda_n)\,(\varphi_n,\,u)\,\varphi_n$. Describe when, if ever, the sequence $(C_N)_{N=1}^{\infty}$ converges strongly in $\mathcal{B}(H)$.

(c) Describe when, if ever, the sequence $(C_N)_{N=1}^{\infty}$ converges in norm in $\mathcal{B}(H)$.

Solution:

(a) If $||A|| < |b|$, then B_N converges in norm, since, as $N \to \infty$,

$$||B_\infty - B_N|| = ||\sum_{n=N+1}^{\infty}\left(\frac{A}{ib}\right)^n|| \leq \sum_{n=N+1}^{\infty}\left(\frac{||A||}{|b|}\right)^n = \left(\frac{||A||}{|b|}\right)^{N+1}\frac{1}{1 - ||A||/|b|} \to 0.$$

Conversely, if $||A|| \geq |b|$, then there exists a λ such that $|\lambda| = ||A||$ and a $v \neq 0$ such that $A\,v = \lambda v$. Then B_N cannot even converge strongly since

$$||B_N v - B_{N-1} v|| = ||-\frac{1}{ib}\frac{\lambda^N}{(ib)^N}\,v|| = \left|\frac{\lambda}{b}\right|^N ||v|| \geq ||v||.$$

(b) C_N always converges strongly. To prove this, we need to show that for any u, $||C_N\,u - C_\infty\,u|| \to 0$. We fix a $u \in H$, and set $u_n = (\varphi_n,\,u)$. Then, as $N \to \infty$,

$$||C_N\,u - C_\infty\,u||^2 = ||\sum_{n=N+1}^{\infty} f(\lambda_n)\,u_n\,\varphi_n||^2 = \sum_{n=N+1}^{\infty}|f(\lambda_n)|^2\,|u_n|^2$$

$$\leq \left(\sup_n |f(\lambda_n)|^2\right)\sum_{n=N+1}^{\infty}|u_n|^2 \leq \frac{1}{|b|^2}\sum_{n=N+1}^{\infty}|u_n|^2 \to 0.$$

The fact that $\sup|f(\lambda_n)|^2 \leq 1/|b|^2$ follows from the fact that all λ_n are real (since A is self-adjoint).

(c) C_N never converges in norm. To prove this, we note that A is compact, so $\lambda_n \to 0$ and $f(\lambda_n) \to -1/ib$. Thus, there exists an M such that $n \geq M$ implies that $|f(\lambda_n)| \geq 1/2|b|$. It follows that for any N, $||C_N - C_\infty|| \geq 1/2|b|$ since for any $m > \max(M, N)$, we have

$$||C_N - C_\infty|| \geq ||(C_N - C_\infty)\varphi_m|| = ||f(\lambda_m)\varphi_m|| = |f(\lambda_m)| \geq \frac{1}{2|b|}.$$

Problem 4: Let R denote a real number such that $0 < R < \infty$ and define

$$f_n(x) = \begin{cases} n\,\cos(nx) & \text{for } |x| \le R, \\ 0, & \text{for } |x| > R. \end{cases}$$

For which numbers R, if any, is it the case that $f_n \to 0$ in $\mathcal{S}^*$?

Solution: $f_n \to 0$ in $\mathcal{S}^*$ if and only if $R = m\,\pi$, for some positive integer m.

To prove this, we recall that $f_n \to 0$ in $\mathcal{S}^*$ if and only if $\langle f_n,\, \varphi\rangle \to 0$ for every $\varphi \in \mathcal{S}$. We have, for any $\varphi \in \mathcal{S}$,

$$\begin{aligned}
\langle f_n,\, \varphi\rangle &= \int_{-R}^{R} n\,\cos(nx)\,\varphi(x)\,dx \\
&= [\sin(nx)\varphi(x)]_{-R}^{R} - \int_{-R}^{R} \sin(nx)\,\varphi'(x)\,dx \\
&= [\sin(nx)\varphi(x)]_{-R}^{R} - \left[-\frac{1}{n}\cos(nx)\varphi'(x)\right]_{-R}^{R} - \int_{-R}^{R} \frac{1}{n}\cos(nx)\,\varphi''(x)\,dx \\
&= \underbrace{\sin(nR)\,\big(\varphi(R)+\varphi(-R)\big)}_{=I_n} + \underbrace{\frac{\cos(nR)}{n}\big(\varphi'(R)-\varphi'(-R)\big)}_{=J_n} + \underbrace{\int_{-R}^{R} \frac{1}{n}\cos(nx)\,\varphi''(x)\,dx}_{=K_n}.
\end{aligned}$$

As $n \to \infty$, we have

$$|J_n| \le \frac{1}{n}\big(|\varphi'(R)| + |\varphi'(-R)|\big) \le \frac{1}{n}2||\varphi||_{1,0} \to 0.$$

Likewise,

$$|K_n| \le \frac{1}{n}\int_{-R}^{R} |\varphi''(x)|\,dx = \frac{1}{n}\int_{-R}^{R} \frac{1}{1+|x|^2}(1+|x|^2)|\varphi''(x)|\,dx \le \frac{1}{n}\pi||\varphi||_{2,2} \to 0.$$

So

$$\begin{aligned}
\langle f_n,\, \varphi\rangle \to 0 \quad &\Leftrightarrow \quad I_n \to 0, \\
&\Leftrightarrow \quad \sin(nR) \to 0, \\
&\Leftrightarrow \quad R = m\,\pi, \quad \text{for } m = 1, 2, 3\ldots
\end{aligned}$$