Homework set 8 — APPM5450

Problem 1: We say that a sequence $(\varphi_n)_{n=1}^{\infty}$ is an *approximate identity* if
(1) $\varphi_n \in C(\mathbb{R}^d), \quad \forall\, n,$
(2) $\varphi_n(x) \geq 0, \quad \forall\, n, x,$
(3) $\int_{\mathbb{R}^d} \varphi_n(x)\, dx = 1, \quad \forall\, n,$
(4) $\forall \varepsilon > 0, \quad \int_{|x|\geq\varepsilon} \varphi_n(x)\, dx \to 0$ as $n \to \infty$.

(a) Do the conditions imply that $\varphi_n \in \mathcal{S}^*$?

(b) Assuming that $\varphi_n \in \mathcal{S}^*$, prove that $\varphi_n \to \delta$ in $\mathcal{S}^*$.

Problem 2: Compute the Fourier transforms of $f(x) = \chi_{[-R,R]}(x)$ and $f(x) = e^{-a|x|}$, *cf.* examples 11.32 and 11.33.

Problem 3: Let k be a positive integer. Prove that there exist numbers c_k and C_k such that $0 < c_k \leq C_k < \infty$, and

$$(1) \qquad c_k\,(1 + |x|^k) \leq (1 + |x|^2)^{k/2} \leq C_k\,(1 + |x|^k), \qquad \forall\, x \in \mathbb{R}^d.$$

Check to see if you can readily adapt your proof to also prove the existence of numbers b_k and B_k such that $0 < b_k \leq B_k < \infty$ such that

$$(2) \qquad b_k\,(1 + |x|)^k \leq (1 + |x|^2)^{k/2} \leq B_k\,(1 + |x|)^k, \qquad \forall\, x \in \mathbb{R}^d.$$

Note 1: The existence of inequalities such as (1) and (2) are routinely used (generally without even commenting on it) to replace the growth factor $(1+|x|^2)^{k/2}$ in the norms $||\cdot||_{\alpha,k}$ by either $(1+|x|^k)$ or $(1+|x|)^k$, whenever convenient.

Note 2: If you have time, you may find it interesting to see what happens to the numbers b_k, B_k, c_k, C_k as k grows large. (This is easily done using Matlab.)

From the textbook: 11.5, 11.9, 11.10.