Homework set 5 — APPM5450, Spring 2006

From the textbook: If you did all of the problems 9.1 – 9.11, then you deserve a rest. If not, do 9.10(b,c,e,g), and 9.11.

Problem 1: Let A be a self-adjoint compact operator. For $\lambda \in \rho(A)$, set $R_\lambda = (A - \lambda I)^{-1}$ as usual. Construct the spectral decomposition of R_λ. Use it to prove that

$$||R_\lambda|| = \text{dist}(\lambda, \sigma(A)) = \inf_{\mu \in \sigma(A)} |\lambda - \mu|.$$

Problem 2: Consider the Hilbert space $H = L^2(I)$, where $I = [-\pi, \pi]$. Define

$$\Omega_t = \{u \in H : u(x) = 0 \; \forall \, x \geq t\}.$$

Note that Ω_t is a closed linear subspace of H. Define $P(t)$ as the orthogonal projection onto Ω_t. Consider the operator $A \in \mathcal{B}(H)$ defined by

$$[Au](x) = x \, u(x).$$

(a) Prove that Ω_t is an invariant subspace of A for every $t \in \mathbb{R}$.

(b) Prove that if $a < b \leq c < d$, then $\text{ran}(P(b) - P(a)) \perp \text{ran}(P(d) - P(c))$. Conclude that for any numbers $-\pi = t_0 < t_1 < t_2 < \cdots < t_n = \pi$, it is the case that

$$H = \text{ran}[P(t_1) - P(t_0)] \oplus \text{ran}[P(t_2) - P(t_1)] \oplus \cdots \oplus \text{ran}[P(t_n) - P(t_{n-1})],$$

where each term is an invariant subspace of A.

(c) For a positive integer n, set $h = 2\pi/n$, and $\lambda_j = -\pi + h\,j$. Define the operator

$$(1) \qquad\qquad A_n = \sum_{j=1}^{n} \lambda_j \left(P(\lambda_j) - P(\lambda_{j-1}) \right).$$

Prove that $||A - A_n|| < 2\pi/n$. Conclude that $A_n \to A$ in norm.

Note: The sum (1) is a Riemann-Stieltjes sum of the integral $A = \int_I \lambda \, dP(\lambda)$.