Homework set 12 — APPM5450, Spring 2006

From the textbook: 12.8, 12.13, 12.14.

Problem 1: Let $(f_n)_{n=1}^{\infty}$ be a sequence of real-valued measurable functions on $\mathbb{R}$ such that $|f_n(x)| \leq 1$ and $\lim_{n\to\infty} f_n(x) = 1$ for all x. Evaluate

$$\lim_{n\to\infty} \int_{\mathbb{R}} f_n(\cos x)\, e^{-\frac{1}{2}(x-2\pi n)^2}\, dx.$$

Make sure to justify your calculation.

Problem 2: Let λ be a real number such that $\lambda \in (0,1)$, and let a and b be two non-negative real numbers. Prove that

$$a^{\lambda}\, b^{1-\lambda} \leq \lambda\, a + (1-\lambda)\, b, \tag{1}$$

with equality iff $a = b$.

Hint: Consider the case $b = 0$ first. When $b \neq 0$, change variables to $t = a/b$.

Problem 3: [Hölder's inequality] Suppose that p is a real number such that $1 < p < \infty$, and let q be such that $p^{-1} + q^{-1} = 1$. Let (X, μ) be a measure space, and suppose that $f \in L^P(X, \mu)$ and $g \in L^q(X, \mu)$. Prove that $fg \in L^1(X, \mu)$, and that

$$||fg||_1 \leq ||f||_p\, ||g||_q. \tag{2}$$

Prove that equality holds iff $\alpha|f|^p = \beta|g|^q$ for some α, β such that $\alpha\beta \neq 1$.

Hint: Consider first the case where $||f||_p = 0$ or $||g||_q = 0$. For the case $||f||_p\, ||g||_q \neq 0$, use (1) with

$$a = \left|\frac{f(x)}{||f||_p}\right|^p, \qquad b = \left|\frac{g(x)}{||g||_q}\right|^q, \qquad \lambda = \frac{1}{p}.$$

Problem 4: [Minkowski's inequality] Let (X, μ) be a measure space, and let p be a real number such that $1 \leq p \leq \infty$. Prove that for $f, g \in L^p(X, \mu)$,

$$||f + g||_p \leq ||f||_p + ||g||_p.$$

Hint: Consider the cases $p = 1, \infty$ separately. For $p \in (1, \infty)$, note that

$$|f(x) + g(x)|^p \leq \big(|f(x)| + |g(x)|\big)\, |f(x) + g(x)|^{p-1}, \qquad \forall\, x \in X. \tag{3}$$

Then integrate both sides of (3) and apply (2) to the right hand side.