APPM5440 — Applied Analysis: Section exam 3

17:00 – 18:15, Nov. 30, 2012. Closed books.

WRITE YOUR NAME: ____

Fill out your answers to problems 1 and 2 directly on the problem sheet. No motivations required.

Write your answers to problems 3 and 4, with motivations, either on the exam, or on separate sheets.

Problem 1: (10p) No motivation required — please just write the answers. 2p per problem.

(a) Let X be any set, and let $\mathcal{T}$ be a collection of subsets of X. Write the conditions that $\mathcal{T}$ must satisfy in order to be a *topology* on X.

(b) For which values of p is the Banach space ℓ^p reflexive? Answer: ____________________________

(c) Let X and Y be normed linear spaces. Mark the true statements:

	Check if true:
For $\mathcal{B}(X,Y)$ to be complete, it is sufficient for X to be complete.	
For $\mathcal{B}(X,Y)$ to be complete, it is sufficient for Y to be complete.	
For $\mathcal{B}(X,Y)$ to be complete, both X and Y must be complete.	

(d) Set $I = [-\pi, \pi]$ and $X = C(I)$. Consider the operator $T \in \mathcal{B}(X)$ defined by

$$[Tf](x) = \int_{-\pi}^{\pi} \sin(x-y)\, f(x)\, dx, \qquad x \in I.$$

Determine the range of T. Answer: $\operatorname{ran}(T) =$ ____________________________

(e) Let c be a real number, let $X = \ell^2$, and define the operator $T \in \mathcal{B}(X)$ via

$$T(x_1,\, x_2,\, x_3,\, x_4,\, \dots) = \left(\left(c + \tfrac{1}{1}\right) x_1,\ \left(c + \tfrac{1}{2}\right) x_2,\ \left(c + \tfrac{1}{3}\right) x_3,\ \left(c + \tfrac{1}{4}\right) x_4,\ \dots\right).$$

For which values of c is the range of T closed?

Solution:

(a) See text book or course notes.

(b) For $p \in (1, \infty)$.

(c) Only the middle one is correct. (Y must be complete, but X does not need to be.)

(d) $\operatorname{ran}(T) = \operatorname{span}\{f, g\}$ where $f(x) = \sin(x)$ and $g(x) = \cos(x)$. (See HW 10: Problem 5.7.)

(e) $\boxed{c \neq 0}$. If $c = 0$, then $\inf \|Tx\|/\|x\| = 0$, so the closed range theorem says that the range is not closed. Set

$$d = \inf_{n=1,2,3,\dots} |c + 1/n|.$$

For instance, if $c > 0$, then $d = c$. You can prove that $\inf \|Tx\|/\|x\| = d$, so if $d > 0$, then the closed range theorem applies and establishes that the range of T is closed. The remaining case is when $c = -1/n$ for some integer n. In this case $d = 0$. However, in this case you can apply the closed range theorem on the subspace of vectors whose n'th coordinate is zero to establish that the range is closed in this case too.

Problem 2: (10p) Set $I = [0, 2]$ and let X denote the space of continuous functions on I. Define a functional on X via $\varphi(f) = \int_0^2 x^2\, f(x)\, dx$.

(a) (5p) Equip X with the norm $\|f\| = \sup_{x \in I} |f(x)|$. Compute $\|\varphi\|_{X^*}$.

(b) (5p) Equip X with the norm $\|f\| = \int_0^2 |f(x)|\, dx$. Compute $\|\varphi\|_{X^*}$.

Solution:

(a) $\boxed{\|\varphi\|_{X^*} = 8/3}$ First we prove an upper bound:

$$\|\varphi\|_{X^*} = \sup_{\|f\|=1} \left| \int_0^2 x^2 f(x)\, dx \right| \leq \sup_{\|f\|=1} \int_0^2 x^2 |f(x)|\, dx \leq \sup_{\|f\|=1} \int_0^2 x^2 \|f\|\, dx = \int_0^2 x^2\, dx = 8/3.$$

For the lower bound, simply use the function $g(x) = 1$:

$$\|\varphi\|_{X^*} = \sup_{\|f\|=1} \left| \int_0^2 x^2 f(x)\, dx \right| \geq \left| \int_0^2 x^2 g(x)\, dx \right| = \int_0^2 x^2\, dx = 8/3.$$

(b) $\boxed{\|\varphi\|_{X^*} = 4}$ For notational convenience, set $h(x) = x^2$. Let $\|\cdot\|_\infty$ denote the sup-norm, and observe that $\|h\|_\infty = f(2) = 4$. Then

$$\|\varphi\|_{X^*} = \sup_{\|f\|=1} \left| \int_0^2 h(x) f(x)\, dx \right| \leq \sup_{\|f\|=1} \int_0^2 |h(x) f(x)|\, dx \leq \sup_{\|f\|=1} \int_0^2 \|h\|_\infty |f(x)|\, dx = \sup_{\|f\|=1} 4\|f\| = 4.$$

For the lower bound, consider the functions

$$g_n(x) = \begin{cases} 0 & \text{for } x \in [0, 2 - 1/n], \\ 2n + 2n^2(x-2) & \text{for } x \in [2 - 1/n, 2]. \end{cases}$$

You can easily verify that $\|g_n\| = 1$. Then

$$\|\varphi\|_{X^*} = \sup_{\|f\|=1} \left| \int_0^2 x^2 f(x)\, dx \right| \geq \sup_n \int_0^2 x^2 g_n(x)\, dx = \sup_n \int_{2-1/n}^2 x^2 g_n(x)\, dx \geq$$

$$\geq \sup_n \int_{2-1/n}^2 (2 - 1/n)^2 g_n(x)\, dx = \sup_n (2 - 1/n)^2 = 4.$$

Problem 3: (10p) Consider the map $f: \mathbb{R}^2 \to \mathbb{R}$ defined by $f((x_1, x_2)) = x_1$. We use the standard Euclidean norm on both $\mathbb{R}^2$ and on $\mathbb{R}$.

(a) (5p) Prove that f is open.

(b) (5p) Prove that f does not necessarily map closed sets to closed sets.

Solution:

(a) Let G be an open set in $\mathbb{R}^2$ and set $H = f(G)$. We need to prove that H is open. Suppose $a \in H$. Then $A \in G$ for some point $A = (a, b)$ in $\mathbb{R}^2$. Since G is open, there is an $\varepsilon > 0$ such that $B_\varepsilon(A) \subseteq G$. But then

$$f(B_\varepsilon(A)) \subseteq f(G) = H.$$

Since $f(B_\varepsilon(A)) = (a - \varepsilon, a + \varepsilon)$, this proves that H is open.

(Alternatively, you can verify that f is onto, continuous, and linear, and then invoke the open mapping theorem.)

(b) Consider the set

$$F = \{(a, b) : \ a \in (0, \infty) \text{ and } b \geq 1/a\}.$$

In other words, F is the area above the curve $f(x) = 1/x$ for $x > 0$. Then F is closed, but

$$f(F) = (0, \infty)$$

is an open set in $\mathbb{R}$.

Problem 4: (10 points) Let X denote a Banach space.

(a) (3p) Let $\{T_n\}_{n=1}^{\infty}$ be a sequence $\mathcal{B}(X)$. Define the following concepts:
 (i) $\{T_n\}$ converges *in norm.*
 (ii) $\{T_n\}$ converges *strongly.*
 (iii) $\{T_n\}$ converges *weakly.*

(b) (5p) Let $X = \ell^1$ with the usual norm. Consider the sequence of operators $\{T_n\}_{n=1}^{\infty}$ defined by

$$T_n(x_1,\, x_2,\, x_3,\, \dots) = (x_1,\, x_2,\, x_3,\, \dots,\, x_{n-1},\, x_n,\, 0,\, 0,\, 0,\, \dots).$$

Does $\{T_n\}$ converge in any of the three modes? Please motivate your answer.

(c) (2p) Let $\{T_n\}_{n=1}^{\infty}$ denote the same operators as in (b), but now acting on $X = \ell^\infty$. Does $\{T_n\}_{n=1}^{\infty}$ converge strongly?

Solution:

(a) See the text book.

(b) (T_n) converges strongly and in norm to the identity operator. It does not converge in norm.
Let e_n denote the n'th canonical basis vector so that $e_1 = (1,0,0,\dots)$, $e_2 = (0,1,0,0,\dots)$, etc. Then

$$\|T_n e_{n+1} - T_{n+1} e_{n+1}\| = \|0 - e_{n+1}\| = 1,$$

so $(T_n)_{n=1}^{\infty}$ is not Cauchy with respect to the operator norm. It can therefore not converge.

Next we prove that (T_n) converges strongly to the identity operator I. Let $x = (x_1,\, x_2,\, x_3,\, \dots) \in X$ be any vector. Then

$$\|T_n x - Ix\| = \|(0,\, 0,\, \dots,\, 0,\, -x_{n+1},\, -x_{n+2},\, -x_{n+3},\, \dots)\| = \sum_{m=n+1}^{\infty} |x_m|.$$

Since $\sum_{n=1}^{\infty} |x_n| < \infty$, it must be the case that $\lim_{n\to\infty} \|T_n x - Ix\| = 0$.

Finally, observe that since $(T_n x)$ converges in norm to x, it must also be the case that $(T_n x)$ converges weakly to x, so (T_n) converges weakly to the identity operator.

(c) (T_n) does not converge strongly. Consider the vector $x = (1,\, 1,\, 1,\, \dots) \in \ell^\infty$. Set

$$y_n = T_n x = (1,\, 1,\, 1,\, \dots,\, 1,\, 0,\, 0,\, \dots) = \sum_{m=1}^{n} e_m, \qquad n = 1,2,3,\dots$$

Then if $m \neq n$, we have $\|y_n - y_m\| = 1$, so (y_n) cannot converge in norm. This proves that (T_n) does *not* converge strongly.