APPM5440 — Applied Analysis: Section exam 1

10:00 – 10:50, Sep. 23, 2016. Closed books.

Important: *Complete problems 1, 2, and 3 in class, and hand your solution in no later than 10:50am. Then complete questions 4 and 5 at home (individual work, no group efforts) and hand the solution in on Wednesday September 28 at the beginning of class at 10:00am.*

Problem 1: Consider the set $X = \mathbb{R}^3$. Let p be a real number such that $0 < p < \infty$.

(a) For which values of p in the interval $(0, \infty)$ is the following function a metric on $X = \mathbb{R}^3$:

$$d(x, y) = \big(|x_1 - y_1|^p + |x_2 - y_2|^p + |x_3 - y_3|^p\big)^{1/p}.$$

(b) For which values of p in the interval $(0, \infty)$ is the following function a metric on $X = \mathbb{R}^3$:

$$d(x, y) = \big(|x_1 - y_1|^p + |x_2 - y_2|^p\big)^{1/p} + |x_3 - y_3|.$$

(c) For which values of p in the interval $(0, \infty)$ is the following function a metric on $X = \mathbb{R}^3$:

$$d(x, y) = |x_1 - y_1|^p + |x_2 - y_2| + |x_3 - y_3|.$$

No motivation is necessary, just write down your answer to each part. Observe carefully that the question is about *metrics,* not *norms.*

Problem 2: Let (X, d) be a metric space, and let Ω be a subset of X.

(a) Define what it means for Ω to be *totally bounded.*

(b) Suppose that X itself is totally bounded. Does X necessarily have a countable dense subset? If you answer yes, then prove this. If you answer no, then provide a counter example.

Problem 3: In this problem, let (X, d) denote a metric space.

(a) Let Ω be a subset of X. State the definition of the *closure* of Ω.

(b) Consider the set of rational numbers $X = \mathbb{Q}$ equipped with the standard metric (the absolute value function). Set $\Omega = \{x \in X : x^2 < 2\}$. What is the closure of Ω?

(c) State the definition of a *completion* of (X, d).

(d) Consider the set X of *positive* rational numbers. What is the completion of X? (The completion is not unique, of course, but there is one very natural candidate.)

Take home exam below.

Problem 4: Set $X = \ell^2$. In other words, an element $x \in X$ if it is a sequence of real numbers $x = (x_1, x_2, x_3, \dots)$ such that $\sum_{n=1}^{\infty} |x_n|^2 < \infty$. The norm on X is $\|x\| = \big(\sum_{n=1}^{\infty} |x_n|^2\big)^{1/2}$. Let B denote the unit ball, so that $B = \{x \in X : \|x\| \leq 1\}$. Prove that B is not a compact set.

Problem 5: Set $I = [0, 1]$ and let X denote the set of real-valued piecewise continuous functions f on I such that

$$\int_0^1 |f(x)|^2 \, dx < \infty.$$

(Since f is piecewise continuous, this is a plain Riemann integral.) Define the function n on X via

$$n(f) = \int_0^1 |f(x)| \, dx.$$

(a) Prove that the function n is a seminorm on X.

(b) Construct a sequence of functions $(f_n)_{n=1}^{\infty}$ in X that is Cauchy with respect to n, and that converges pointwise to a function on I that does not belong to X.