Homework set 13 — APPM5440 — Fall 2016

Problem 1: Let X be a normed linear space, let M be a closed subspace, and let $\hat{x}$ be an element not contained in M. Set

$$d = \text{dist}(M, \hat{x}) = \inf_{y \in M} ||y - \hat{x}||.$$

Prove that $d > 0$. Prove that there exists an element $\varphi \in X^*$ such that $\varphi(\hat{x}) = 1$, $\varphi(y) = 0$ for $y \in M$, and $||\varphi|| = 1/d$.

Hint: Set $Z = \text{Span}(M, \hat{x})$. Prove that any $z \in Z$ can be written $z = y + \alpha\,\hat{x}$ for a unique $\alpha \in \mathbb{R}$ and a unique vector $y \in M$. Define ψ as a suitable functional on Z, and then extend it to X using the Hahn-Banach theorem.

Solution:

First we prove that $d > 0$. Suppose M is a closed linear subspace, and that x is a point such that $\text{dist}(M, x) = 0$. Then there are $x_n \in M$ such that $\lim ||x_n - x|| = 0$. Since M is closed and $x_n \to x$, we must have $x \in M$. Since $\hat{x} \notin M$, it follows that $d > 0$.

Set $Z = \text{Span}(M, \hat{x})$.

Prove that any $z \in Z$ can be written $z = y + \alpha\,\hat{x}$ for a unique $\alpha \in \mathbb{R}$ and a unique vector $y \in M$. (This is not hard.)

Define for $z \in Z$ the functional ψ via $\psi(z) = \alpha$, where α is the unique number such that $z = y + \alpha\hat{x}$. Then $\psi(\hat{x}) = 1$ and $\psi(y) = 0$ for every $y \in M$.

We will now prove that the norm of ψ viewed as a functional on Z equals $1/d$. To this end, set

$$C = \sup_{z \in Z,\ z \neq 0} \frac{|\varphi(z)|}{||z||}.$$

We then need to prove that $C = 1/d$. First observe that for any $z \in Z \backslash M$ we have

$$||z|| = ||y + \alpha\hat{x}|| = |\alpha|\,||\frac{1}{\alpha}y + \hat{x}|| \geq |\alpha|\,d.$$

(Observe that $||\frac{1}{\alpha}y + \hat{x}|| \geq d$ since $(1/\alpha)y \in M$ and the distance between any element in M and $\hat{x}$ is at least d.) It follows that

$$|\varphi(z)| = |\alpha| \leq \frac{||z||}{d}.$$

This shows that $C \leq 1/d$. To prove the opposite inequality, pick $y_n \in M$ such that

$$\lim_{n \to \infty} ||\hat{x} - y_n|| = d.$$

Set $z_n = \hat{x} - y_n$. Then

$$C \geq \lim_{n \to \infty} \frac{|\varphi(z_n)|}{||z_n||} = \lim_{n \to \infty} \frac{1}{||z_n||} = \frac{1}{d}.$$

Finally, invoke the Hahn-Banach to assert the existence of an extension of ψ to all of X satisfying all requirements.

Problem 2: Let X be a normed linear space with a linear subspace M. Prove that the weak closure of M equals the closure of M in the norm topology. *Hint:* Use Problem 3.

Solution:

Since the norm closure of any set is contained in the weak closure, all we need to prove is that any point *not* in the norm closure is also not in the weak closure.

Suppose $\hat{x} \notin \bar{M}$. From Problem 3, we know that there exists a functional $\varphi \in X^*$ such that $\varphi(\hat{x} - y) = 1$ for any vector $y \in \bar{M}$. Since M is a subset of $\bar{M}$, this shows that there can be no sequence in M that converges weakly to $\hat{x}$.

Problem 3: Prove that the following statements follow from the Hahn-Banach theorem:

(a) For any $x \in X$, there is a $\varphi \in X^*$ such that $||\varphi|| = 1$ and $\varphi(x) = ||x||$.

(b) For any $x \in X$, $||x|| = \sup_{||\varphi||=1} |\varphi(x)|$.

(c) If $x, y \in X$ and $x \neq y$, there is a $\varphi \in X^*$ such that $\varphi(x) \neq \varphi(y)$.

(d) For $x \in X$, define $F_x \in X^{**}$ by setting $F_x(\varphi) = \varphi(x)$.
Prove that the map $x \mapsto F_x$ is a linear isometry from X to X^{**}.

Solution:

. . . see class notes . . .

Problem 4: (Lax equivalence) Let X and Y be Banach spaces, let $A \in \mathcal{B}(X, Y)$ be an operator with a continuous inverse, let $f \in Y$, and consider the equation

$$A\,u = f.$$

Now suppose that we have "some mechanism" for approximating the equation to any given precision. In other words, given $\varepsilon > 0$, we can construct A_ε that approximates A, and f_ε that approximates f, and such that the equation

$$A_\varepsilon\,u_\varepsilon = f_\varepsilon$$

can be solved. (Typically, A_ε is a finite dimensional operator, so that the approximate equation can be solved by solving a finite system of linear algebraic equations.) We say that

- The approximation is *consistent* if $A_\varepsilon \to A$ strongly.
- The approximation is *stable* if there is an $M < \infty$ such that $||A_\varepsilon^{-1}|| \leq M$ for all $\varepsilon > 0$.
- The approximation is *convergent* if $u_\varepsilon \to u$ whenever $f_\varepsilon \to f$ (in norm).

Suppose that the approximation scheme is consistent. Prove that then:

$$\text{The scheme is convergent} \quad \Leftrightarrow \quad \text{The scheme is stable}$$

Hint: The solution is in the text book, but please try it yourself before looking!

Note: In practice, variations of this result are often used in the context of approximating partial differential equations via, e.g., finite elements or finite differences. In this case, the operator is not bounded — this assumption can be done away with.

Solution:

$\boxed{\Rightarrow}$ Assume that the scheme is stable. Then

$$u - u_\varepsilon = A_\varepsilon^{-1}(A_\varepsilon u - f_\varepsilon) = A_\varepsilon^{-1}(A_\varepsilon u - Au + f - f_\varepsilon).$$

Consequently,

$$\|u - u_\varepsilon\| \leq \|A_\varepsilon^{-1}\|\big(\|A_\varepsilon u - Au\| + \|f - f_\varepsilon\|\big) \to 0$$

as $\varepsilon \to 0$.

$\boxed{\Leftarrow}$ Assume that A_ε is not stable. We will build two sequences of vectors (u_n) and (f_n) such that $f_n \to 0$ in norm, but $u_n = A_{\varepsilon_n}^{-1} f_n$ does not converge to zero, where (ε_n) is a sequence converging to zero. (In other words, we construct approximations to the solution $u = 0$ of $Au = 0$.)

Since A_ε is not stable, there is a sequence of unit vectors (v_n) and a sequence of (ε_n) such that $\lim_n \varepsilon_n = 0$ and $\|A_{\varepsilon_n}^{-1} v_n\| \to \infty$.

Define

$$f_n = \frac{v_n}{\|A_{\varepsilon_n}^{-1} v_n\|}.$$

Then $\|f_n\| \to 0$, so (f_n) does indeed converge in norm to 0. Moreover, set $u_n = A_{\varepsilon_n}^{-1} f_n$, so that u_n is the solution to $A_{\varepsilon_n} u_n = f_n$. Then (with $u = 0$, of course) we have

$$\|u - u_n\| = \|u_n\| = \|A_{\varepsilon_n}^{-1} f_n\| = \left\| \frac{A_{\varepsilon_n}^{-1} v_n}{\|A_{\varepsilon_n}^{-1} v_n\|} \right\| = 1.$$

In other words, the discretization is not convergent.