Homework 1 — partial solutions — APPM5440, Fall 2016

Problem 1.3: Note that the desired inequality is equivalent to the following pair of inequalities:

$$\begin{cases} d(x,z) - d(y,z) \leq d(x,y) \\ d(y,z) - d(x,z) \leq d(x,y) \end{cases}$$

Now prove each of the two inequalities in the pair above separately.

Problem 1.5: We will prove that if $(X, \|\cdot\|)$ is a NLS, then the function

$$d(x,y) = \frac{\|x-y\|}{1+\|x-y\|}$$

defines a metric on X. It is easy to verify that d is symmetric and is zero iff $x = y$. The challenge is the triangle inequality. Observe that

$$d(x,y) = f(\|x-y\|), \qquad \text{where} \qquad f(t) = \frac{t}{1+t}.$$

Since f is monotonically increasing, and since $\|x-y\| \leq \|x-z\| + \|y-z\|$, we immediately find that

$$d(x,y) = f(\|x-y\|) \leq f(\|x-z\| + \|y-z\|).$$

Next, use the following lemma:

Lemma: Suppose that $f : [0,\infty) \to [0,\infty)$ is a differentiable function that satisfies $f(0) = 0$, $f' \geq 0$, and f' is monotonically decreasing. Then $f(a+b) \leq f(a) + f(b)$ for every non-negative a and b.

Proof: We have

$$f(a+b) = f(a) + \int_a^{a+b} f'(t)\,dt \leq f(a) + \int_0^b f'(t)\,dt = f(a) + f(b) - f(0) = f(a) + f(b),$$

where the inequality holds true since f' is positive but decreasing (so for every t we have $f'(t) \leq f'(a+t)$.)

Since our f satisfies this property, we immediately get

$$d(x,y) = f(\|x-y\|) \leq f(\|x-z\| + \|y-z\|) \leq f(\|x-z\|) + f(\|y-z\|) = d(x,z) + d(z,y).$$

Problem 2: (a) The putative norms a, d, e, and f are norms. (b and g are semi-norms, c does not satisfy $||\alpha f|| = |\alpha|\,||f||$.)

(c) Set $I = [0,\, 1]$ and consider the set X consisting of all continuous functions on I, with the norm

$$||f|| = \int_0^1 |f(x)|\,dx.$$

Prove that the space X is not complete.

Solution: A straight-forward way of proving this is to construct a Cauchy-sequence that does not have a limit point in X. One example is

$$f_n(x) = \begin{cases} -1 & x < 1/2 - 1/n, \\ n(x-1/2) & 1/2 - 1/n \leq x \leq 1/2 + 1/n, \\ 1 & x > 1/2 + 1/n. \end{cases}$$

We first prove that (f_n) is Cauchy. Note that for any m, n, and x, we have $|f_n(x) - f_m(x)| \leq 1$. When $m, n \geq N$, we further have $f_n(x) - f_m(x) = 0$ outside the interval $[1/2 - 1/N, 1/2 + 1/N]$, so

$$||f_n - f_m|| = \int_{1/2-1/N}^{1/2+1/N} |f_n(x) - f_m(x)|\, dx \leq \int_{1/2-1/N}^{1/2+1/N} 1\, dx = 2/N.$$

We next prove that (f_n) cannot converge to any element in X. Pick an arbitrary $\varphi \in X$. Assume temporarily that $\varphi(1/2) \geq 0$. Since φ is continuous, there exists a $\delta > 0$ such that $\varphi(x) \geq -1/2$ for $x \in B_\delta(1/2)$. Pick an integer $N > 2/\delta$. Then, for $n \geq N$, we have $f_n(x) = -1$ when $x \in [1/2 - \delta, 1/2 - \delta/2]$, and so

$$||f_n - \varphi|| \geq \int_{1/2-\delta}^{1/2-\delta/2} |f_n(x) - \varphi(x)|\, dx \geq \int_{1/2-\delta}^{1/2-\delta/2} 1/2\, dx = \delta/4.$$

If on the other hand $\varphi(1/2) < 0$, then pick $\delta > 0$ such that $\varphi(x) \leq 1/2$ on $[1/2, 1/2 + \delta]$ and proceed analogously. $\square$

Remark 1: Note that you cannot solve a problem like the one above by constructing a Cauchy sequence (f_n) in X, point to a non-continuous function f, and claim that since f_n "converges to f", X cannot be complete. Note that the metric is *not even defined* for functions outside of X.

Remark 2: Can you somehow add the limit points of Cauchy sequences in X and obtain a complete space $\tilde{X}$? The answer is yes, you can do that for any metric space; the resulting space $\tilde{X}$ is called the "completion" of X and is (in a certain sense) unique. For the present example, $\tilde{X}$ is the set of all (Lebesgue measurable) real-valued functions on I for which

$$\int_0^1 |f(x)|\, dx < \infty,$$

where the integral is what is called a "Lebesgue" integral. This space is denoted $L^1(I)$. Strictly speaking, an element of $L^1(I)$ is an equivalence class of functions that differ only on a set of Lebesgue measure zero. This roughly means that two functions f and g are considered identical if

$$\int_0^1 |f(x) - g(x)|\, dx = 0.$$