Applied Analysis (APPM 5440): Final exam

7:30pm – 10:00pm, Dec. 11, 2016. Closed books.

Problem 1: (16p) No motivations required for these problems. 4p each.

(a) Let X be a set, and let $\mathcal{T}$ denote a topology on X. Define what it means for $\mathcal{T}$ to satisfy the Hausdorff property.

See textbook.

(b) Let X denote a Banach space. Mark the following statements as true/false:

	TRUE	FALSE
If $(T_n)_{n=1}^{\infty}$ is a sequence in $\mathcal{B}(X)$ of compact operators that converges in norm to an operator T, then T is necessarily compact.	TRUE	
Let $S, T \in \mathcal{B}(X)$. If S is compact, then ST is compact.	TRUE	
Let $S, T \in \mathcal{B}(X)$. If S is compact, then TS is compact.	TRUE	
Let $S, T \in \mathcal{B}(X)$. If S and T are both compact, then $S + T$ is compact.	TRUE	
Let $S, T \in \mathcal{B}(X)$. If S is compact, then $S + T$ is compact.		FALSE

(c) Set $I = [0, 1]$ and $X = C(I)$. (We use the standard norm on X.) Define the subset

$$A = \{u \in X : u \text{ is continuously differentiable and } \|u'\| \leq 1\}.$$

Describe the closure $\bar{A}$ of A:

The set Lipschitz continuous functions f such that Lip$(f) \leq 1$.

Is $\bar{A}$ a compact set (yes/no)? *No. (The set is not bounded.)*

(d) Set $H = L^2([-1, 1])$, and define $T \in \mathcal{B}(H)$ via $[Tu](x) = 2u(-x)$. Let $S \in \mathcal{B}(H)$ be an operator for which you know that $\|S\| \leq c$, where c is some positive number. Are there any values of c for which you can say for sure that the operator $T - S$ has closed range?

$\boxed{c < 2}$ *(Observe that T is invertible, so $T - S = T(I - T^{-1}S)$. If $\|T^{-1}S\| < 1$, then the Neumann formula tells us that $T - S$ is invertible as well, which implies that its range is all of H, which is closed. Since $\|T^{-1}\| = 1/2$, we see that $\|T^{-1}S\| < 1$ if $\|S\| < 2$.*

Problem 2: (16p) Let H denote a Hilbert space. Prove that for every element $\varphi \in H^*$, there exists a unique $y \in H$ such that

$$\varphi(x) = (y, x), \qquad \forall x \in H.$$

Solution:

See the course notes for a clean an short proof that works in any Hilbert speace.

Many solutions attempted to use the fact that every HS has an ON basis to build the vector y. This can be done, but is more work, and uses more machinery than is necessary. For completeness, let us see how a correct argument along these lines would work.

First observe that if H is finite dimensional, then things are straight-forward. Let $\{e_i\}_{i=1}^n$ denote an ON basis. Then any vector $x \in H$ admits an expansion $x = \sum_{i=1}^n x_i e_i$, so

$$\varphi(x) = \varphi\left(\sum_{i=1}^n x_i e_i\right) = \sum_{i=1}^n x_i \varphi(e_i).$$

Now define

$$y = \sum_{i=1}^n \overline{\varphi(e_i)}\, e_i$$

(observe the complex conjugate!) to see that

$$\varphi(x) = \sum_{i=1}^n \varphi(e_i) x_i = \sum_{i=1}^n \overline{(e_i, y)}\,(e_i, x) = (y, x).$$

Next let us consider the general case where H has a basis $\{e_\alpha\}_{\alpha\in A}$. Superficially, the same argument can be applied here, but you have to prove that $\sum_{\alpha\in A} |\varphi(e_\alpha)|^2 < \infty$ so that the formula for y actually defines a vector in H. To this end, let B denote any finite subset of A and set

$$z = \frac{1}{\left(\sum_{\alpha\in B} |\varphi(e_\alpha)|^2\right)^{1/2}} \sum_{\alpha\in B} \overline{\varphi(e_\alpha)}\, e_\alpha.$$

Then $\|z\| = 1$, so we find that

$$\begin{aligned}\|\varphi\|_{H^*} \geq |\varphi(z)| = &\left| \frac{1}{\left(\sum_{\alpha\in B} |\varphi(e_\alpha)|^2\right)^{1/2}} \sum_{\alpha\in B} \overline{\varphi(e_\alpha)}\, \varphi(e_\alpha)\right| \\ &= \left| \frac{1}{\left(\sum_{\alpha\in B} |\varphi(e_\alpha)|^2\right)^{1/2}} \sum_{\alpha\in B} |\varphi(e_\alpha)|^2\right| = \left(\sum_{\alpha\in B} |\varphi(e_\alpha)|^2\right)^{1/2}.\end{aligned}$$

Finally take the sup over all finite $B \subseteq A$ to get

$$\|\varphi\|_{H^*} \geq \left(\sum_{\alpha\in A} |\varphi(e_\alpha)|^2\right)^{1/2}.$$

Note: However you built y, you lost points if you did not prove uniqueness.

Problem 3: (16p) Set $I = [0, \pi]$ and let H denote the Hilbert space $H = L^2(I)$ with the usual norm. Define $f, g, h \in H$ via

$$f(x) = \sin(x), \qquad g(x) = \sin(3x), \qquad h(x) = x.$$

Set $N = \text{Span}\{f, g\}$, and $M = N^{\perp}$. Evaluate

$$d = \inf_{u \in M} \|h - u\|.$$

In the event that you make any computational errors, your score on this problem will depend strongly on whether you clearly present the argument on how you determine d.

Solution:

First observe that $H = M \oplus N$, so the vector h can be uniquely decomposed as $h = m + n$ with $m \in M$ and $n \in N$. Then $d = \|n\|$ since $n = h - m$, where m is the closest point in M to h.

In order to determine n, we will build an orthonormal basis for N. Simple calculations show that

$$\int_0^{\pi} |f(x)|^2\,dx = \frac{\pi}{2}, \qquad \int_0^{\pi} |g(x)|^2\,dx = \frac{\pi}{2}, \qquad \int_0^{\pi} f(x)g(x)\,dx = 0.$$

Consequently, an ON basis for H is given by the two vectors

$$u_1(x) = \beta\,\sin(x), \qquad u_2(x) = \beta\,\sin(3x), \qquad \text{where} \qquad \beta = \sqrt{2/\pi}.$$

The $n = (u_1, h)u_1 + (u_2, h)u_2$. The coefficients are easily determined via partial integration:

$$(u_1, h) = \beta \int_0^{\pi} x\,\sin(x)\,dx = \beta\,[-x\,\cos(x)]_0^{\pi} + \beta \int_0^{\pi} \cos(x)\,dx = \beta\pi + 0,$$

$$(u_2, h) = \beta \int_0^{\pi} x\,\sin(3x)\,dx = \beta\,[-x\,\cos(3x)/3]_0^{\pi} + \beta \int_0^{\pi} \cos(3x)/3\,dx = \beta\pi/3 + 0.$$

Finally, we get

$$d = \|n\| = \sqrt{|(u_1, h)|^2 + |(u_2, h)|^2} = \sqrt{\beta^2\pi^2 + \beta^2\pi^2/9} = \sqrt{2\pi + 2\pi/9} = \sqrt{20\pi/9} = 2\sqrt{5\pi}/3.$$

Note: Very few answers identified d correctly. Forgetting to normalize the basis vectors for N was particularly common. However, you got a healthy amount of points as long as you described a correct basic idea. The key observation I looked for was that $d = \|n\|$ where n is the orthogonal projection onto N (not onto M!). Observe that there is no need in this problem to describe M in any detail, or to build an ON basis for M.

Problem 4: (16p) Set $I = [0, 2]$, set $X = C(I)$, and let k be a continuous function on $I \times I$. Consider the operator $T \in \mathcal{B}(X)$ defined by

$$[Tu](x) = \int_0^2 k(x, y)\, u(y)\, dy, \qquad x \in I.$$

(a) State the Arzelá-Ascoli theorem.

(b) Prove that the operator T is compact.

Solution:

(a) See the text book.

(b) We will prove that T is compact by showing that it maps any bounded set to a pre-compact set. Let B be a bounded set in X. Set $M = \sup\{\|u\| : u \in B\}$. We will prove that TB is bounded and equicontinuous. Then, since I is compact, the AA theorem asserts that TB is pre-compact and we will be done.

Proof that TB is bounded: Set $C = \sup\{|k(x, y)| : (x, y) \in I \times I\}$. Since k is continuous, and $I \times I$ is compact, we know that C is finite. Then for any $u \in B$, we have

$$\|Tu\| = \sup_{x \in I} \left| \int_0^2 k(x, y) u(y)\, dy \right| \leq \sup_{x \in I} \int_0^2 |k(x, y)|\, |u(y)|\, dy \leq \sup_{x \in I} \int_0^2 C\, M\, dy = 2CM.$$

Proof that TB is equicontinuous: Fix $\varepsilon > 0$. Since k is continuous on the compact set $I \times I$, there is a $\delta > 0$ such that for every $y \in I$, we have

$$|x - z| < \delta, \qquad \Rightarrow \qquad |k(x, y) - k(z, y)| < \varepsilon/(2M).$$

Suppose $|x - z| < \delta$. Then for any $v \in TB$, let $u \in B$ be such that $v = Tu$. Then

$$|v(x) - v(z)| = \left| \int_0^2 \big(k(x, y) - k(z, y)\big)\, u(y)\, dy \right| \leq \int_0^2 \big|k(x, y) - k(z, y)\big|\, |u(y)|\, dy < \int_0^2 \frac{\varepsilon}{2M} M\, dy = \varepsilon.$$

Note: Some solutions did not include a proof that T is bounded. Since this fact was listed in the problem formulation, and since I did not explicitly ask you to prove it, I did not deduct any points for this omission.

Some solutions to (b) used an incorrect definition of a compact operator. If you used a definition that sidesteps the compactness part, you got zero points. Beside the definition used in the solution above, the other one that is convenient is that T is compact if the image of any bounded sequence has a convergent subsequence.

Problem 5: (16p) Let X denote the space of all continuous functions on $\mathbb{R}$ that are periodic with period 1. In other words, if $u \in X$, then

$$u(x) = u(x+1), \qquad \forall x \in \mathbb{R}.$$

We equip X with the norm

$$\|u\| = \sup_{x\in[0,1]} |u(x)|.$$

Observe that a function u in X is uniquely defined by its values on the interval $I = [0,1]$ (or on $[0,1)$, for that matter, since $u(0) = u(1)$). Define for $n = 1, 2, 3, \dots$ the operators

$$[T_n u](x) = u(x - 1/n).$$

(a) (6p) Does $(T_n)_{n=1}^{\infty}$ converge strongly? Please motivate your answer carefully.
(b) (6p) Does $(T_n)_{n=1}^{\infty}$ converge in norm? Please motivate your answer carefully.
(c) (4p) Do your answers change if X is instead equipped with the norm $\|u\| = \int_0^1 |u(x)|\,dx$?

Solution:

(a) We will prove that (T_n) converges strongly to the identity operator I. Fix $u \in X$, and pick any $\varepsilon > 0$. Since u is a continuous function on the compact set $[-1,1]$ (for instance), we know that u is uniformly continuous on this interval. Consequently, there is a $\delta > 0$ such that

$$|x - y| < \delta \qquad \Rightarrow \qquad |u(x) - u(y)| < \varepsilon.$$

Suppose that $n > 1/\delta$. Then

$$\|u - T_n u\| = \sup_{x\in[0,1]} |u(x) - u(x - 1/n)| \leq \{\text{Use that} |x - (x - 1/n)| = 1/n < \delta\} \leq \sup_{x\in[0,1]} \varepsilon = \varepsilon.$$

(b) Since $T_n \to I$ strongly, the only possible point that (T_n) could converge to in norm is I. We will prove that $\|T_n - I\| \geq 1$ for every n, which shows that (T_n) does not converge in norm. Define for $n = 1, 2, 3, \dots$ the functions

$$\psi_n = \begin{cases} 1 - 3n|x|, & \text{for } |x| < 1/(3n), \\ 0, & \text{for } |x| \geq 1/(3n). \end{cases}$$

and

$$u_n(x) = \sum_{n=-\infty}^{\infty} \psi_n(x - n).$$

Then $\|u_n\| = 1$ and $u_n \in X$. Moreover,

$$\|I - T_n\| \geq \|u_n - T_n u_n\| = \sup_{x\in[0,1]} |u_n(x) - u_n(x - 1/n)| \geq |u_n(0) - u_n(-1/n)| = |1 - 0| = 1.$$

(c) The answers remain the same. For strong convergence, note that if $T_n u \to u$ uniformly, then it is necessarily the case that $\int_0^1 |u - T_n u|\,dx \to 0$. To prove that (T_n) does not converge in norm, an analogous argument works if you define ψ_n as in the solution to (b), and then define u_n via

$$u_n(x) = \sum_{n=-\infty}^{\infty} 3n\psi_n(x - n).$$

Then $\|u_n\| = 1$, and $\|I - T_n\| \geq \|u_n - T_n u_n\| = 2$.

Note: In proving part (a), the uniform continuity of u is important. Many solutions had a simple claim that $\|u - T_n u\| = \sup_{x\in[0,1]} |u(x) - u(x - 1/n)| \to 0$. If no motivation was given for this step, you lost 2 points.