THE "WEAK-STAR" TOPOLOGY

Let X be a Banach space, and consider the space X^*. We are by now familiar with two types of convergence in X^*:

(*) $\varphi_n \to \varphi$ IN NORM if $\|\varphi - \varphi_n\| \to 0$ as $n \to \infty$.

(*) $\varphi_n \to \varphi$ WEAKLY if $F(\varphi_n) \to F(\varphi)$ $\forall F \in X^{**}$.

We next define a third mode of convergence:

(*) $\varphi_n \to \varphi$ WEAK-* if $\varphi_n(x) \to \varphi(x)$ $\forall x \in X$

$\Leftrightarrow F_x(\varphi_n) \to F_x(\varphi)$ $\forall x \in X$.

Since $X \subseteq X^{**}$, the weak-* topology is a weaker (or equivalent) topology to the weak topology.

If X is reflexive, $X^{**} = X$, then the weak and the weak-* topologies are the same.

The Hahn-Banach theorem implies that the weak-* topology is Hausdorff.

The weak-* topology is useful because the unit ball is typically compact in this top.

To be precise, set: $S^* = \{\varphi \in X^* : \|\varphi\| \leq 1\}$. Then

S^* is compact in the norm top $\Leftrightarrow$ X is finite dim.

S^* is compact in the weak top $\Leftrightarrow$ X is reflexive

S^* is <u>always</u> compact in the weak-* top!

Thm (Alaoglu) Let $\mathfrak{X}$ be a NLS.
Let S^* denote the closed unit ball in $\mathfrak{X}^*$.
Then S^* is a compact Hausdorff space in the weak-* topology.

Thm Let $\mathfrak{X}$ be a Banach space and let S denote its closed unit ball. Then:
S is compact in the weak top $\iff$ $\mathfrak{X}$ is reflexive.