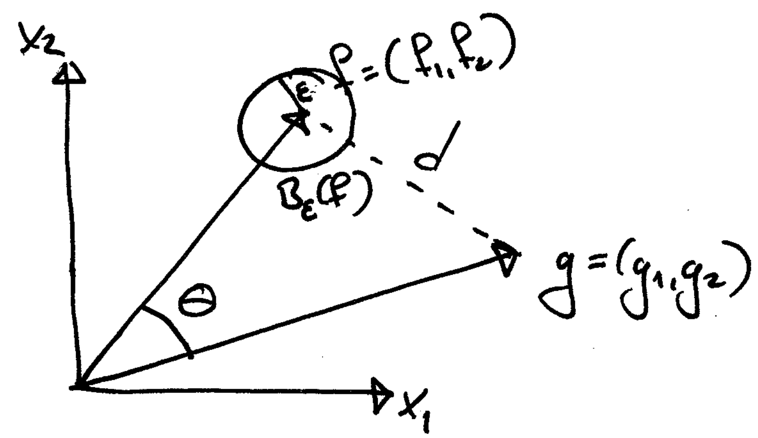
INTRO LECTURE

Explain how geometric concepts from $\mathbb{R}^d$ can be generalized to ~~$\mathbb{R}^d$~~ infinite dimensional spaces, in particular to function spaces.
Mention both similarities and the fundamentally new phenomena that arise in infinite dimensional spaces.

Recall $\mathbb{R}^d$:

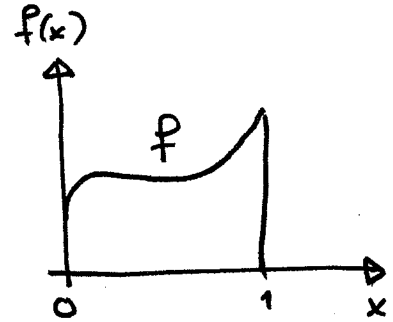


$$d = d(f,g) = \sqrt{(f-g,\, f-g)} = \sqrt{\sum_{i=1}^{d} |f_i - g_i|^2}$$

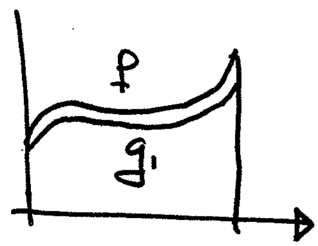
$$\cos\theta = \frac{(f,g)}{\|f\|\;\|g\|} \qquad \|f\| = d(f,0)$$

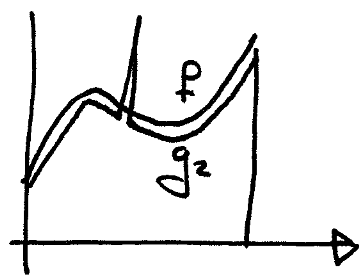
In contrast, in a function space, "vectors" are functions.

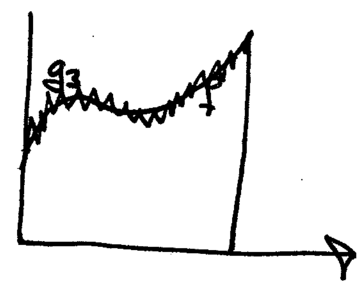
Example $I = [0,1]$ $\quad X =$ "functions from I to $\mathbb{R}$"
(this has to be put in precise terms later!)



$f \in X$ is a <u>point</u> in X.
What functions are "close" to f?







Example $L^2(I)$ = functions on I s.t. $\int |f(x)|^2 dx < \infty$

$(f,g) = \int f(x) g(x) dx \qquad \|f\| = \left(\int |f(x)|^2 dx \right)^{1/2}$

g_1, g_2, g_3 are all close to f.

Example $p \in [1, \infty)$, $\quad L^p(I)$ = functions s.t. $\left(\int |f(x)|^p \right)^{1/p} < \infty$

No inner product.

Local deviations penalized if p is large

Extreme case is $L^\infty(I)$, $\quad \|f\| = \sup_{x \in I} |f(x)|$

Example $H^1(I)$ $\qquad (f,g) = \int \left(f(x) g(x) + f'(x) g'(x) \right) dx$

Only g_1 is close to f!

What is the derivative here?

Must functions in H^1 be continuous? C^1?

Example $C(I)$ - closely related to $L^\infty(I)$.
but $C(I)$ has only continuous functions.
$\|f\| = \sup_x |f(x)|$

$C^1(I)$: $\|f\| = \sup(|f(x)| + |f'(x)|)$
What is relationship between C^1 & H^1?
Can these norms be derived from an inner product?

We will study a hierarchy of more and more general spaces:

* Euclidean space $\mathbb{R}^d$: $(f,g) = \sum_{j=1}^{d} f_j g_j$
* Inner product spaces (Hilbert spaces) $L^2(I)$ $H^1(I)$ $(f,g) = \int_\Omega f(x)g(x)dx$

$\|f\| = \sqrt{(f,f)}$
$d(f,g) = \|f-g\|$
~~$\|f\| = \sqrt{(f,g)}$~~

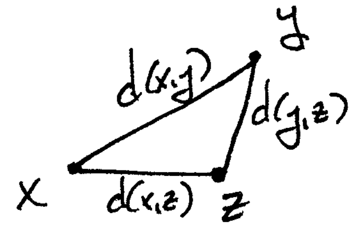
* Normed linear spaces (Banach spaces) $L^p(I)$ $C(I)$ $C^1(I)$ $d(f,g) = \|f-g\|$
no inner product
* Metric spaces - distance is not necessarily given by a norm.
Examples: * Riemannian manifolds (curved spaces)
Think of a sphere, with $d(f,g)$ = shortest "flying distance" between f ~~and~~ and g.
* Schwartz space $d(f,g) = \sum_{n=0}^{\infty} \sum_{j=0}^{n} \sup_{x\in\mathbb{R}} |x|^j \left| \left(\frac{\partial}{\partial x}\right)^{n-j} (f-g)(x) \right|$
* Topological spaces - no distance function.
Wave hands...

Metric Spaces

Def^n Let $\mathcal{X}$ be a non-empty set, and suppose that d is a function

$$d : \mathcal{X} \times \mathcal{X} \to [0, \infty)$$

such that



(i) $d(x,y) = 0 \iff x = y$

(ii) $d(x,y) = d(y,x) \quad \forall x,y \in \mathcal{X}$

(iii) $d(x,y) \le d(x,z) + d(z,y) \quad \forall\, x,y,z \in \mathcal{X}$

Then $(\mathcal{X}, d)$ is called a METRIC SPACE with the METRIC d.

Example $\mathbb{R}^n$ with $d(x,y) = \|x - y\|$

Example $\mathbb{R}^n$ with $d(x,y) = \max\limits_{1 \le j \le n} |x_j - y_j| \qquad x = (x_1, x_2, \dots, x_n)$

Proof (i) & (ii) are obvious.

For (iii) we find

$$d(x,y) = \max_{1 \le j \le n} |x_j - y_j| = \max |(x_j - z_j) + (z_j - y_j)| \le$$

$$\le \max\left(|x_j - z_j| + |z_j - y_j|\right) \le \max |x_j - z_j| + \max |z_j - y_j|$$

$$= d(x,z) + d(z,y)$$

Example Let $\mathcal{X}$ be any set.

Define $d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \ne y \end{cases}$

Claim Let (X, d) be a metric space.
Let Y be a subset of X.
Then (Y, d) is a metric space.
We call (Y, d) a <u>subspace</u> of (X, d).

Example $X = \mathbb{R}$ $\quad d(x,y) = |x-y|$
Then $(\mathbb{Q}, d)$ is a subspace of $(\mathbb{R}, d)$

Def^n Let F be a scalar field ($\mathbb{R}$ or $\mathbb{C}$).
Then a set X is called a VECTOR FIELD over F (or a LINEAR SPACE) if there exist operations "+" and "·" such that

X is a commutative group w.r.t. +
$$\left\{\begin{array}{ll}
(i) & x+y = y+x \quad \forall x,y \in X \\
(ii) & (x+y)+z = x+(y+z) \quad \forall x,y,z \in X \\
(iii) & \exists\, 0 \in X \text{ such that } x+0 = x \quad \forall x \in X \\
(iv) & \forall x \in X \ \exists \text{ an element } "-x" \in X \text{ s.t. } x+(-x) = 0
\end{array}\right.$$

Conditions on scalar multiplication
$$\left\{\begin{array}{lll}
(v) & 1 \cdot x = x & \forall x \in X \\
(vi) & (\lambda+\mu)x = \lambda x + \mu x & \forall \lambda,\mu \in F \quad x \in X \\
(vii) & \lambda(\mu x) = (\lambda\mu)x & \text{---} \ " \ \text{---} \\
(viii) & \lambda(x+y) = \lambda x + \lambda y & \forall \lambda \in F \quad x,y \in X
\end{array}\right.$$

Def^n A linear space X (over F) is called a NORMED LINEAR SPACE if there exists a map $\|\cdot\| : X \to [0,\infty)$ such that

(i) $\|\lambda x\| = |\lambda| \, \|x\|$

(ii) $\|x+y\| \le \|x\| + \|y\|$

(iii) $\|x\| = 0 \iff x = 0$

Claim A normed linear space is a metric space with the metric $d(x,y) = \|x-y\|$.

Example $X = \mathbb{R}^n$ $\quad \|x\|_p = \left(\sum_{j=1}^{d} |x_j|^p \right)^{1/p}$ for $p<\infty$

$\|x\|_\infty = \cancel{\sup} \max_{1\le j\le n} |x_j| \; (= \lim_{p\to\infty} \|x\|_p)$

Claim $(X, \|\cdot\|_p)$ is a NLS when $1 \le p \le \infty$

Proof (i) & (iii) are trivial

(ii) is a little bit of work unless $p = 1, 2, \infty$

Homework!

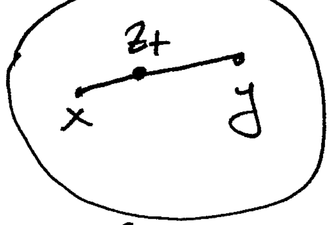
What about the case $p<1$? Let's take a detour:

Def^n Let A be a subset of a NLS X.

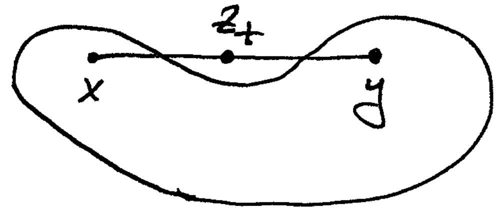
We say that A is CONVEX if $\forall x,y \in A$ & $t \in [0,1]$,

$z_t = tx + (1-t)y \in A$

Examples



Convex



Not convex

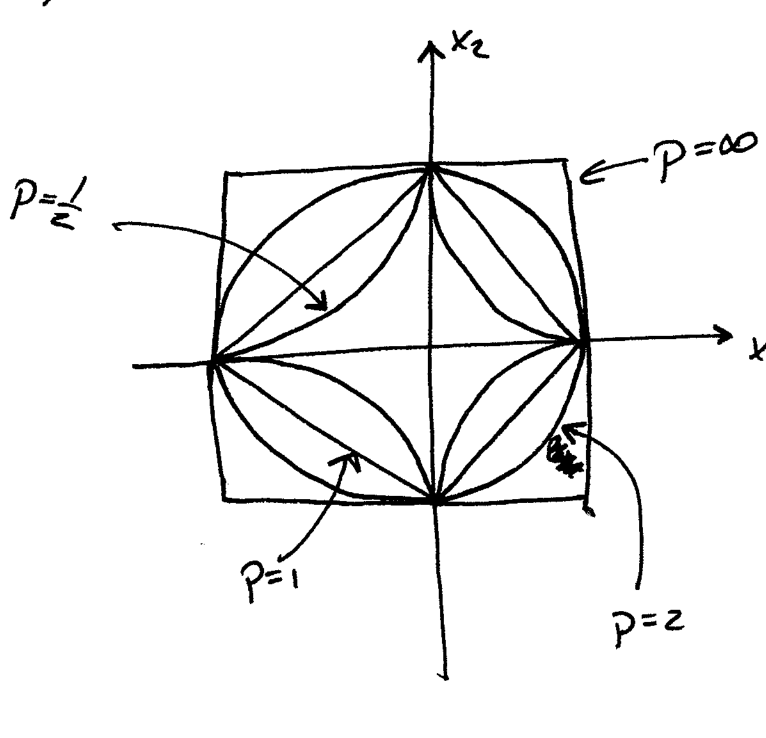
Lemma Let $\mathbb{X}$ be a NLS and set
$B = \{x \in \mathbb{X} : \|x\| \leq 1\}$ ← "Closed unit ball"
Then B is convex

Proof Pick $x, y \in B$ and $t \in [0,1]$. Then
$\|z_t\| = \|tx + (1-t)y\| \leq \|tx\| + \|(1-t)y\| = t\|x\| + (1-t)\|y\| \leq 1$
So $z_t \in B$.

Claim $(\mathbb{R}^n, \|\cdot\|_p)$ is NOT a NLS ~~wen~~ when $p<1$.

Proof We will prove that the unit ball is not convex.
Set $e^{(1)} = [1,0,\dots,0]$, $e^{(2)} = [0,1,0,\dots,0]$, then $e^{(1)} \& e^{(2)} \in B$.
$z_{1/2} = [\frac{1}{2}, \frac{1}{2}, 0, 0, \dots, 0]$.
$\|z_{1/2}\|_p = \left(\frac{1}{2^p} + \frac{1}{2^p}\right)^{1/p} = (2\, 2^{-p})^{1/p} = 2^{\frac{1}{p}-1} > 1$ if $p<1$
so $z_{1/2} \notin B$.

In $\mathbb{R}^2$, we find that the unit ball has the following shapes:



CONVERGENCE, COMPLETENESS

Def^n Let (X,d) be a metric space, and let $(x_n)_{n=1}^\infty$ be a sequence in X.

* We say that $x_n \to x$ if $d(x_n, x) \to 0$. CONVERGENCE
* We say that (x_n) is CAUCHY if $\forall \varepsilon > 0\ \exists N$ such that $m,n \geq N \Rightarrow d(x_n, x_m) < \varepsilon$
* We say that X is COMPLETE if every Cauchy sequence in X has a limit point in X.

Note: Every convergent sequence is necessarily Cauchy.

Thm: $\mathbb{R}^n$ is a complete metric space. ← Important.

Example $X = (0,1)$. Is X complete? No: $x_n = \frac{1}{n}$ is Cauchy.

Example Every closed subset of $\mathbb{R}^n$ is a complete metric space. (We will return to the ~~a~~ question of when a set is "closed" shortly.)

Example $\mathbb{Q}$ with the usual metric is not complete. For a counterexample, pick for any integer n a rational ~~numb~~ number $q_n \in (\sqrt{2} - \frac{1}{n}, \sqrt{2} + \frac{1}{n})$. Then $(q_n)_{n=1}^\infty$ is Cauchy, but ~~not~~ does not converge to a point in $\mathbb{Q}$.

UPPER & LOWER BOUNDS

Def^n Let A be a subset of $\mathbb{R}$.

- If M is a number such that $x \le M \quad \forall x \in A$, then M is an UPPER BOUND of A.
- If M is an upper bound of A, and there is no smaller upper bound then we say that M is the *LEAST UPPER BOUND of A, $M = \sup A$
- If A does not have an upper bound, set $\sup A = \infty$
- Analogously, define LOWER BOUND, GREATEST LOWER BOUND, $\inf A$
- for the empty set, define $\sup \emptyset = -\infty$
- If $M = \sup A \in A$, set $M = \max A$

Thm Every subset of the real numbers has a unique supremum and infimum in the reals.

Note that this is not true for max/min.
Nor is it true for subsets of $\mathbb{Q}$, example: $\Omega = \{q \in \mathbb{Q} : q^2 < 2\}$

Def^n A sequence $(x_n)_{n=1}^{\infty} \subseteq \mathbb{R}$ is MONOTONE INCREASING if $x_n \le x_{n+1} \ \forall n$

—————— " —————— DECREASING if $x_n \ge x_{n+1} \ \forall n$

Every monotone sequence "converges" (possibly to $\pm\infty$),

if (x_n) is monotone increasing, then $\lim_{n\to\infty} x_n = \sup\{x_n\}_{n=1}^{\infty}$

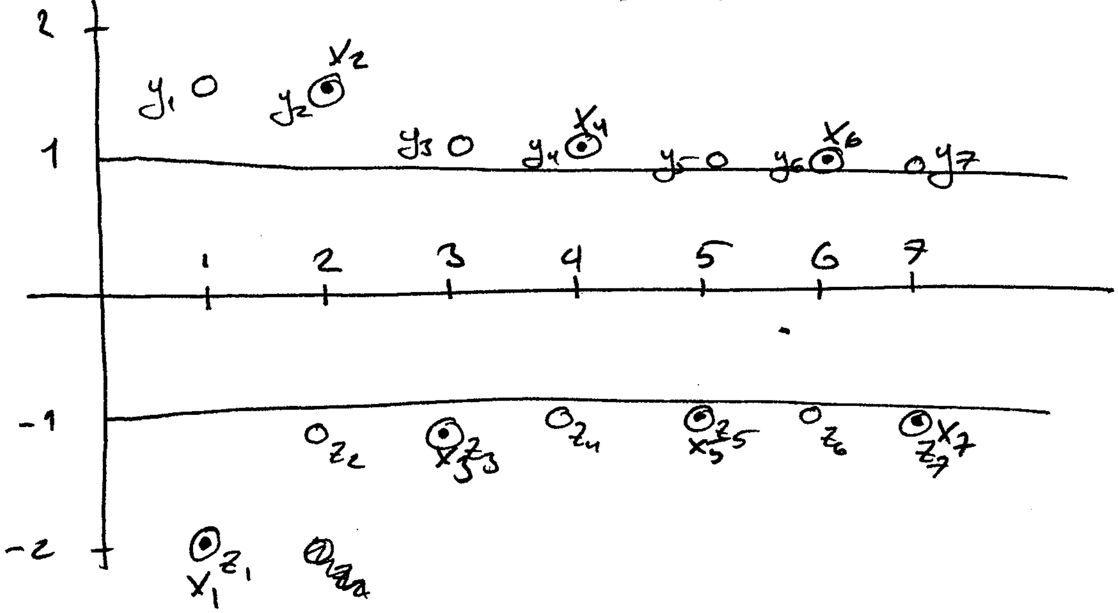
Not every sequence has a limit.
However, every sequence has what we call limsup & liminf.

Given a sequence $(x_n)_{n=1}^{\infty}$, set $y_n = \sup\{x_k : k \geq n\}$
$z_n = \inf\{x_k : k \geq n\}$

Then y_n is monotone decreasing $\Rightarrow$ $\lim y_n$ exists set $\lim y_n = \limsup$

z_n is monotone increasing $\Rightarrow$ $\lim z_n$ exists. Set $\limsup x_n = \lim z_n$

Example: $x_n = (-1)^n \left(1 + \frac{1}{n}\right)$



$\limsup\limits_{n\to\infty} x_n = \lim y_n = 1$ $\qquad$ $\liminf x_n = \lim z_n = -1$

Equivalent definitions:

$$\limsup_{n\to\infty} x_n = \lim_{n\to\infty} \sup\{x_k : k \geq n\}$$

$$= \inf\{\alpha : \text{Only finitely many } x_n \text{ are larger than } \alpha\}$$

$$= \sup\{\alpha : \text{there exists a subseq } (n_j)_{j=1}^{\infty} \text{ such that } \alpha = \lim_{j\to\infty} x_{n_j}\}$$

Lemma For *any* sequence $(x_n)_{n=1}^{\infty}$ ~~in ℝ~~ in $\mathbb{R}$:

* $\limsup x_n$ and $\liminf x$ exist (but may equal $\pm\infty$)
* $\liminf x_n \leq \limsup x_n$
* (x_n) is convergent $\Leftrightarrow$ $\limsup x_n = \liminf x_n$

  In this case, $\lim x_n = \limsup x_n = \liminf x_n$, of course.

CONTINUITY

Def^n Let X and Y be metric spaces, and suppose $f: X \to Y$.

* We say that f is CONTINUOUS at x if $\forall \varepsilon > 0$, $\exists \delta > 0_{\varepsilon}$ such that $d_X(x, x') < \delta \Rightarrow d_Y(f(x), f(x')) < \varepsilon$.
* We say that f is CONTINUOUS if it is cont. at every $x \in X$.
* We say that f is SEQUENTIALLY CONTINUOUS AT x if for every seq. (x_n) s.t. $x_n \to x$, we have $f(x_n) \to f(x)$
* We say that f is "sequentially continuous" if f is seq cont. at every $x \in X$.

Note: We will soon prove that seq. cont $\Leftrightarrow$ cont.

Example Let X and Y be metric spaces.

Suppose that X has the discret metric, $d_X(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$

Then *any* function $f: X \to Y$ is seq. cont

Proof: Suppose $x_n \to x$ in X. We need to prove that $f(x_n) \to f(x)$ in Y.

But if $x_n \to x$ in X, $\exists N$ s.t. $n \geq N \Rightarrow x_n = x$.

Then for $n \geq N$, $f(x_n) = f(x)$ so $f(x_n) \to f(x)$.

OPEN & CLOSED SETS

Def^n Let (X, d) be a metric space.

* For $c \in X$, $r \in [0, \infty)$, define $B_r(c) = \{x \in X : d(x,c) < r\}$ ← "open ball"
* ~~for $c \in X$, $r \in [0,\infty)$, define $\overline{B_r(c)} = \{x \in X : d(x,c) \leq$~~
* A set $G \subseteq X$ is OPEN if $\forall\, x \in G$ $\exists\, r > 0$ s.t. $B_r(x) \subseteq G$
* A set $F \subseteq X$ is CLOSED if $F^c = X \setminus F$ is open.
* The BOUNDARY of a set $\Omega \subseteq X$ is the set of all points $x \in X$ such that for any $\varepsilon > 0$, $B_\varepsilon(x)$ contains points in both Ω and Ω^c.

Prop^n Let (X, d) be a metric space.

(i) X and $\emptyset$ are both open and closed.
(ii) A finite intersection of open sets is open.
(iii) Any union of open sets is open.
(iv) A finite union of closed sets is closed.
(v) Any intersection of closed sets is closed.

Note that the requirements of finiteness above is necessary.

$$\bigcap_{n=1}^{\infty} \left(-\tfrac{1}{n}, 1\right) = [0, 1)$$

$$\bigcap_{n=1}^{\infty} \left(-\tfrac{1}{n}, 1 + \tfrac{1}{n}\right) = [0, 1]$$

$$\bigcup_{n=1}^{\infty} \left[\tfrac{1}{n}, 1 - \tfrac{1}{n}\right] = (0, 1)$$

Proof of (iii) Let $\{G_\alpha\}_{\alpha\in A}$ be a collection of open sets and set $G = \bigcup_{\alpha\in A} G_\alpha$.
Fix an $x \in G$. $\exists\, \beta \in A$ s.t. $x \in G_\beta$.
Since G_β is open, $\exists\, \varepsilon > 0$ s.t. $B_\varepsilon(x) \subseteq G_\beta$.
Since $G_\beta \subseteq G$, it follows that $B_\varepsilon(x) \subseteq G$.

Proof of (v): Let $\{F_\alpha\}_{\alpha\in A}$ be a collection of closed sets and set $F = \bigcap_{\alpha\in A} F_\alpha$.
That F is closed follows immediately from (iii) since $F^c = \bigcup_{\alpha\in A} F_\alpha^c$ and all F_α^c are open.

Def^n Let $\mathbb{X}$ & $\mathbb{Y}$ be metric spaces. $f: \mathbb{X} \to \mathbb{Y}$ is "open-set-continuous" if G open in $\mathbb{Y}$ $\Rightarrow$ $f^{-1}(G)$ open in $\mathbb{X}$.

Prop^n ~~Def^n~~ Let $\mathbb{X}$ & $\mathbb{Y}$ be metric spaces, and let f map $\mathbb{X}$ to $\mathbb{Y}$.
TFAE: (a)* f is ε-δ-cont.
(b)* f is seq. cont.
(c) f is open-set cont.

Proof: We will prove that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a)

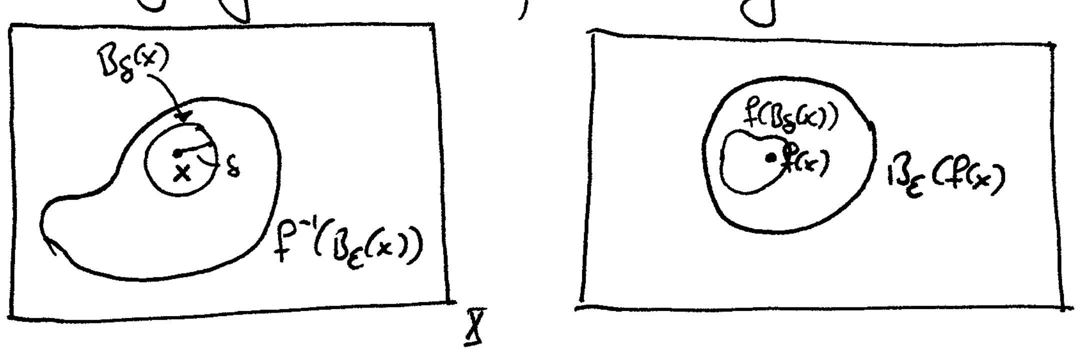
(a)$\Rightarrow$(b) Suppose that f is ε-δ-cont.
We need to prove that if $x_n \to x$, then $f(x_n) \to f(x)$.
Suppose that $x_n \to x$. Fix $\varepsilon > 0$.
f is ε-δ cont at x $\Rightarrow$ $\exists\, \delta$ s.t. $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$.
Since $x_n \to x$, $\exists\, N$ s.t. $n \geq N \Rightarrow x_n \in B_\delta(x)$.
Then for $n \geq N$, $f(x_n) \in B_\varepsilon(f(x))$.

Note: The prop^n says that in a metric space, the three notions of continuity are equivalent. Henceforth, we will simply say "continuous".

(b) ⇒ (c) Suppose that f is NOT open-set cont.
We will construct a seq (x_n) s.t. $x_n \to x$ for some x, but $f(x_n) \not\to f(x)$.
Since f is not open-set cont, $\exists$ an open set $G \subseteq Y$ s.t. $H = f^{-1}(G)$ is not open.
Since H is not open, $\exists x \in H$ s.t. $B_\varepsilon(x) \cap H^c$ is non-empty for every $\varepsilon > 0$.
For $n = 1, 2, 3, \dots$ pick $x_n \in B_{1/n}(x) \cap H^c$.
Then $x_n \to x$.
However, since $f(x) \in G$, and G is open, $\exists \varepsilon$ s.t. $B_\varepsilon(f(x)) \in G$.
Since $f(x_n) \notin G$, it follows that $d(f(x_n), f(x)) > \varepsilon$ $\forall n$ and so $f(x_n)$ cannot converge to $f(x)$.

(c) ⇒ (a) Suppose that f is open-set cont.
We will prove that f is ε-δ-cont. Fix an $x \in X$.
Pick any $\varepsilon > 0$.
Since $B_\varepsilon(f(x))$ is open in Y, $f^{-1}(B_\varepsilon(f(x)))$ is open in X.
Since $x \in f^{-1}(B_\varepsilon(f(x)))$, $\exists \delta > 0$ s.t. $B_\delta(x) \subseteq f^{-1}(B_\varepsilon(f(x)))$.
But then $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$.

__Note__: Drawing figures for the proof is a good idea. For instance, (c) ⇒ (a)



Def^n Suppose that Ω is a subset of a metric space X.
The closure of Ω is the set $\overline{\Omega} = \{x \in X : \exists (x_n) \subset \Omega \text{ s.t. } x_n \to x\}$.

Claim Letting $\partial\Omega$ denote the boundary of Ω, we have $\overline{\Omega} = \Omega \cup \partial\Omega$.

Claim $\overline{\Omega}$ = the intersection of all closed sets containing Ω.

Claim Ω is closed $\iff \Omega = \overline{\Omega}$.

Example Let X denote the metric space $\mathbb{R}$ with the usual metric.
Let $\Omega = \mathbb{Q}$. Then $\overline{\Omega} = \mathbb{R} = X$.

Example Let X denote the set $\mathbb{R}$ and let $d(x,y) = \begin{cases} 0 & \text{if } x=y \\ 1 & \text{if } x \neq y. \end{cases}$
Let Ω be any subset of X. Then $\overline{\Omega} = \Omega$.
(Note that in the discrete metric, ANY set is closed.)

Def^n Let X be a metric space.
A subset Ω is said to be dense in X if $\overline{\Omega} = X$.

~~Example Under It~~

Def^n If a metric space has a dense countable subset,
then we say that the metric space is seperable.

Example $\mathbb{R}$ equipped with the usual metric is seperable since $\mathbb{Q}$ is dense.

Example $\mathbb{R}$ equipped with the discrete metric is NOT seperable since
no ~~sup~~ subset is dense (except $\mathbb{R}$ itself).

Example Let $\mathfrak{X}$ denote the set of all sequences $x=(x_1,x_2,x_3,\ldots)$ such that (i) $x_n\in\mathbb{Q}$ $\forall n$, and (ii) only finitely many x_n's are non-zero. Define for $x=(x_1,x_2,\ldots)$ & $y=(y_1,y_2,y_3,\ldots)$ the metric $d(x,y)=\left(\sum_{n=1}^{\infty}|x_n-y_n|^2\right)^{1/2}$.

~~Then $\mathfrak{X}$ is~~

Note that $\mathfrak{X}$ is countable (to see this, note that $\mathfrak{X}=\bigcup_{n=1}^{\infty}\mathfrak{X}_n$, where $\mathfrak{X}_n$ is the set of all sequences $(x_1,x_2,\ldots,x_n,0,0,0,\ldots)$ where $x_j\in\mathbb{Q}$ $\forall j$, then each $\mathfrak{X}_n$ is countable since it can be identified with $\mathbb{Q}^n$, and thus $\mathfrak{X}$ is a countable union of countable sets).

Next ~~set~~ let $\tilde{\mathfrak{X}}$ denote the set of all sequences $x=(x_1,x_2,\ldots)$ such that $x_n\in\mathbb{R}$ and $\sum_{n=1}^{\infty}|x_n|^2<\infty$. Equip $\tilde{\mathfrak{X}}$ with the same metric as $\mathfrak{X}$.

~~Note that~~ $\mathfrak{X}$ is dense in $\tilde{\mathfrak{X}}$. To prove this, fix any $\underset{x=(x_1,x_2,\ldots)}{x}\in\tilde{\mathfrak{X}}$.
Fix a $\overbrace{j}^{\text{set } \varepsilon=\frac{1}{j}}$. Pick N_j s.t. $\sum_{n=N_j+1}^{\infty}|x_n|^2<\frac{\varepsilon^2}{2}$.

Next, pick for $n=1,2,\ldots,N_j$ numbers $x_n^{(j)}\in\mathbb{Q}$ such that $|x_n-x_n^{(j)}|<\frac{\varepsilon}{\sqrt{2N_j}}$, and set $x^{(j)}=(x_1^{(j)},\ldots,x_{N_j}^{(j)},0,0,\ldots)$.
Then $x^{(j)}\in\mathfrak{X}$, and $d(x,x^{(j)})=\Big(\underbrace{\sum_{n=1}^{N_j}|x_n-x_n^{(j)}|^2}_{<\varepsilon^2/2}+\underbrace{\sum_{n=N_j+1}^{\infty}|x_n|^2}_{<\varepsilon^2/2}\Big)^{1/2}<\varepsilon=\frac{1}{j}$

Thus $x^{(j)}\to x$. ~~and so x~~

COMPLETION OF A METRIC SPACE

It sometimes happen that we define a set X and a metric d and find that the resulting space (X, d) is not complete.
This is highly inconvenient.
It turns out to be possible to "add the missing elements" and obtain a new space $(\tilde{X}, \tilde{d})$ that is complete.
This space is called the COMPLETION of (X, d).
- It is in a certain sense unique.

Example: The set of real numbers $\mathbb{R}$ can be defined by first defining the rational numbers $\mathbb{Q}$ and then form the completion of $\mathbb{Q}$ w.r.t. the metric $d(x,y) = |x-y|$.

Example: The spaces X & $\tilde{X}$ on page 13.

Example: The homework problem where $I = [0,1]$ and
X = the space of all continuous functions on I,
and $d(f,g) = \left(\int_I |f(x) - g(x)|^2 dx \right)^{1/2}$.
Then $\tilde{X} = L^2(I)$ = the space of all Lebesgue-measurable functions f s.t. $\int_0^1 |f(x)|^2 dx < \infty$.

Caveat: In $L^2(I)$, two functions f and g are considered identical if $\int |f(x) - g(x)|^2 dx = 0$.
To be precise, an element in $L^2(I)$ is an equivalence class of functions.

Def^n Let X and Y be metric spaces.
A map $i: X \to Y$ is called an isometry if
$$d_X(x,y) = d_Y(i(x), i(y)) \quad \forall x,y \in X.$$
If is also ONTO, then i is ~~an~~ a metric space isomorphism.

Note 1: Any isometry is necessarily one-to-one
(~~Ex~~ If $i(x)=i(y)$, then $d(x,y)=d(i(x),i(y))=0 \Rightarrow x=y$.)

Note 2: If two spaces X and Y are isomorphic, i.e. if there exists an isomorphism from X to Y, then mathematically speaking, X and Y are identical, they differ only in how the elements are labelled.
The concept of an isomorphism can be defined for metric space, NLS, inner-product spaces, Lie-algebras, etc, etc.
In each case, the isomorphism must preserve all the relevant structure.

Example $\mathbb{R}^2$ and $\mathbb{C}$ are isomorphic metric spaces $i: \mathbb{R}^2 \to \mathbb{C}: \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto x_1 + i x_2$

Example $X = L^2(\underbrace{I}_{I=[-\pi,\pi]},\mathbb{R})$ = the space of "all" real-valued functions s.t. $\int |f|^2 < \infty$.
$Y = l^2(\mathbb{Z},\mathbb{C})$ = the space of all sequences $(\dots, \alpha_{-2}, \alpha_{-1}, \alpha_0, \alpha_1, \alpha_2, \dots)$
such that $\sum_{\infty} |\alpha_j|^2 < \infty$, $\alpha_j \in \mathbb{C}$.
$$d(\alpha,\beta) = \left(\sum_{j=-\infty}^{\infty} |\alpha_j - \beta_j|^2 \right)^{1/2}$$
The Fourier transform $F: X \to Y$ is ~~an isometry~~ defined by
$$F: f \mapsto (\dots, \alpha_{-1}, \alpha_0, \alpha_1, \dots) \quad \text{where } \alpha_n = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{-inx} f(x)\,dx$$

Parseval's equality states that $\int |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |\alpha_n|^2$.

It follows that F is an isometry.

However f is not an isomorphism from $L^2(I, \mathbb{R})$ to $l^2(\mathbb{Z}, \mathbb{C})$ since, if $(\alpha_n) = Ff$, then $\overline{\alpha_n} = \alpha_{-n}$ (in other words, F is not onto).

However, F is an isomorphism from $L^2(I, \mathbb{C})$ - the set of all Lebesgue measurable square-integrable complex-valued functions to $l^2(\mathbb{Z}, \mathbb{C})$. Its invers is given by

$$[F^{-1}(\alpha_n)](x) = \sum_{n=-\infty}^{\infty} \alpha_n \frac{e^{inx}}{\sqrt{2\pi}}.$$

Def^n A metric space $(\tilde{X}, \tilde{d})$ is a completion of the metric space (X, d) if

(a) There is an isometric embedding $i : X \to \tilde{X}$.

(b) $i(X)$ is dense in $\tilde{X}$.

(c) The space $(\tilde{X}, \tilde{d})$ is complete.

Thm Every metric space (X, d) has a completion $(\tilde{X}, \tilde{d})$. If $(\tilde{X}_1, \tilde{d}_1)$ and $(\tilde{X}_2, \tilde{d}_2)$ are both completions of (X, d), then $(\tilde{X}_1, \tilde{d}_1)$ and $(\tilde{X}_2, \tilde{d}_2)$ are isomorphic.

In order to prove the thm, we need the following lemma:

Lemma Let $(x_n)_{n=1}^{\infty}$ be a Cauchy seq. Then (x_n) has a subsequence $(y_j)_{j=1}^{\infty}$ s.t. $y_j = x_{n_j}$ and $m, n \geq N \Rightarrow d(y_n, y_m) \leq \frac{1}{N}$

Proof Let n_1 be s.t. $m, n \geq n_1 \Rightarrow d(x_m, x_n) \leq \frac{1}{1}$

Let n_2 be s.t. $m, n \geq n_2 \Rightarrow d(x_m, x_n) \leq \frac{1}{2}$ (and $n_2 \geq n_1$)

Let n_3 be s.t. $m, n \geq n_3 \Rightarrow d(x_m, x_n) \leq 1/3$ (and $n_3 \geq n_2$)

Etc

Then $y_j = x_{n_j}$ satisfies the criteria.

Proof of thm

Step 1: ~~[illegible]~~ Prove uniqueness

Let $(\tilde{X}_1, \tilde{d}_1)$ & $(\tilde{X}_2, \tilde{d}_2)$ be completions, with isometries i_1 & i_2.

First we define an isometry $\tilde{i} : \tilde{X}_1 \to \tilde{X}_2$

$$\begin{tikzcd} & (\tilde{X}_1, \tilde{d}_1) \arrow[d, "\tilde{i}"] \\ (X, d) \arrow[ur, "i_1"] \arrow[dr, "i_2"'] & \\ & (\tilde{X}_2, \tilde{d})_2 \end{tikzcd}$$

For $\tilde{x} \in \tilde{X}_1$, pick $x_n \in X$ s.t. $i_1(x_n) \to \tilde{x}$ (possible since $i_1(X)$ is dense in $\tilde{X}_1$)

$(i_1(x_n))$ is Cauchy $\underset{\substack{\uparrow \\ i_1 \text{ is an isometry}}}{\Rightarrow}$ (x_n) is Cauchy $\underset{\substack{\uparrow \\ i_2 \text{ is an isometry}}}{\Rightarrow}$ $(i_2(x_n))$ is Cauchy

Since $\tilde{X}_2$ is complete $\exists \tilde{y} \in \tilde{X}_2$ s.t. $i_2(x_n) \to \tilde{y}$.

The element $\tilde{y}$ does not depend on the choice of sequence (x_n);

in fact, if $i_1(z_n) \to \tilde{x}$, then $\tilde{d}_2(i_2(z_n), i_2(x_n)) = \tilde{d}_1(i_1(z_n), i_1(x_n)) \to 0$

so $i_2(z) \to \tilde{y}$ as well. Therefore, we can define $\tilde{i}(\tilde{x}) = \tilde{y}$.

* We need to prove that $\tilde{i}$ is an isometry:

Pick $\tilde{x}, \tilde{y} \in \tilde{X}_1$ and $x_n, y_n \in X$ s.t. $i_1(x_n) \to \tilde{x}$ & $i_1(y_n) \to \tilde{y}$. Then

$\tilde{d}_1(\tilde{x}, \tilde{y}) = \lim \tilde{d}_1(i_1(x_n), i_1(y_n)) = \lim d(x_n, y_n) = \lim \tilde{d}_2(i_2(x_n), i_2(y_n)) = \tilde{d}_2(\tilde{i}(\tilde{x}), \tilde{i}(\tilde{y}))$.

* We need to prove that $\tilde{i}(i_1(X)) = i_2(X)$.

Simply use constant sequences!

* We need to prove that $\tilde{i}$ is ONTO.

Fix any $\hat{x} \in \tilde{X}_2$. Since $i_2(X)$ is dense in $\tilde{X}_2$, $\exists x_n \in X$ s.t. $i_2(x_n) \to \hat{x}$. Then $(i_2(x_n))$ is Cauchy in $\tilde{X}_2 \Rightarrow$

$\Rightarrow (x_n)$ is Cauchy in $X \Rightarrow (i_1(x_n))$ is Cauchy in $\tilde{X}_1$.

Since $\tilde{X}_1$ is complete, $\exists \tilde{x} \in \tilde{X}_1$ s.t. $i_1(x_n) \to \tilde{x}$.

But then $\tilde{i}(\tilde{x}) = \hat{x}$ by def^n.

Step 2: Construct the completion

We define $\tilde{X}$ as a set of equivalence classes on the set of all Cauchy sequences on X.

Suppose that (x_n) & (y_n) are Cauchy seq^s in X.

We say that $(x_n) \sim (y_n)$ ~~id~~ if $\lim_{n\to\infty} d(x_n, y_n) = 0$ (this limit always exists → HOMEWORK)

This equivalence relation defines a set of equivalence classes which we label $\tilde{X}$. Given an equivalence class $\tilde{x} \in \tilde{X}$, we say that a seq $(x_n) \in \tilde{x}$ is a <u>representative</u> of $\tilde{x}$.

* <u>Def^n of $\tilde{d}$</u>: Given $\tilde{x}, \tilde{y} \in \tilde{X}$, pick $(x_n) \in \tilde{x}$ & $(y_n) \in \tilde{y}$ and set $\tilde{d}(\tilde{x}, \tilde{y}) = \lim_{n\to\infty} d(x_n, y_n)$

We need to prove that $\tilde{d}(\tilde{x}, \tilde{y})$ does not depend on the choice of representative.

Suppose $(x_n') \in \tilde{x}$ & $(y_n') \in \tilde{y}$.

Then $d(x_n, y_n) \le \underbrace{d(x_n, x_n')}_{\to 0} + d(x_n', y_n') + \underbrace{d(y_n', y_n)}_{\to 0}$

so $\lim_{n\to\infty} d(x_n, y_n) \le \lim_{n\to\infty} d(x_n', y_n')$

Analogously, $\lim d(x_n', y_n') \le \lim d(x_n, y_n)$.

The proof that $\tilde{d}$ is a metric is left as a homework.

* <u>Construct the embedding $i : X \to \tilde{X}$.</u> Given an $x \in X$, set for $n = 1, 2, 3, \ldots$ $x_n = x$. Then (x_n) is a Cauchy seq.

Define $i(x)$ as the equivalence class containing this seq.

Step 3: Prove that $i(X)$ is dense in $\tilde{X}$

Fix any $\tilde{x} \in \tilde{X}$.

Pick a representative $(x_n) \in \tilde{x}$.

For $n = 1, 2, 3, \ldots$ let $\tilde{x}^{(n)}$ denote the equivalence class that contains the constant sequence ~~$(x_1^{(n)}, x_2^{(n)}$~~ $(x_n, x_n, x_n, \ldots)$.

Then clearly, $\tilde{x}^{(n)} \in i(X)$.

Moreover, $\lim_{n\to\infty} \tilde{d}(\tilde{x}^{(n)}, \tilde{x}) = \lim_{n\to\infty} \lim_{j\to\infty} d(x_j^{(n)}, x_j) = \lim_{n\to\infty} \lim_{j\to\infty} d(x_n, x_j) = 0$,

so $\tilde{x}^{(n)} \to \tilde{x}$ in $\tilde{X}$.

Step 4 - Prove that $\tilde X$ is complete

Let $(\tilde x^{(n)})$ be a Cauchy seq in $\tilde X$.

Let $(\tilde y^{(n)})$ be a subseq of $(\tilde x^{(n)})$ s.t. $\tilde d(\tilde y^{(n)},\tilde y^{(m)}) \le \frac{1}{N}$ when $m,n \ge N$

For each $n = 1,2,3,\dots$ pick a representative $(y_k^{(n)})_{k=1}^{\infty} \in \tilde y^{(n)}$.

We can pick the representative so that $d(y_k^{(n)}, y_l^{(n)}) \le \frac{1}{N}$ when $k,l \ge N$.

Then $(y_k^{(k)})_{k=1}^{\infty}$ is a Cauchy seq in X. ~~since~~

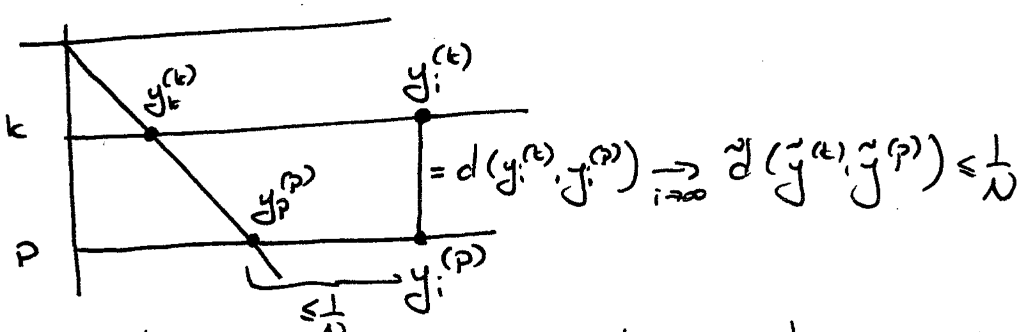
To prove this we first note that for any $i = 1,2,\dots$

$$d(y_k^{(k)}, y_p^{(p)}) \le d(y_k^{(k)}, y_i^{(k)}) + d(y_i^{(k)}, y_i^{(p)}) + d(y_i^{(p)}, y_p^{(p)})$$

Suppose that $k,p \ge N$ and take the limit as $i \to \infty$:

$$d(y_k^{(k)}, y_p^{(p)}) \le \underbrace{\lim_{i\to\infty} d(y_k^{(k)}, y_i^{(k)})}_{\le \frac{1}{N}} + \underbrace{\lim_{i\to\infty} d(y_i^{(k)}, y_i^{(p)})}_{= \underbrace{\tilde d(\tilde y^{(k)}, \tilde y^{(p)})}_{\le \frac{1}{N}}} + \underbrace{\lim_{i\to\infty} d(y_i^{(p)}, y_p^{(p)})}_{\le \frac{1}{N}} \le \frac{3}{N}$$

so $k,p \ge N \Rightarrow d(y_k^{(k)}, y_p^{(p)}) \le \frac{3}{N}$ which means that $(y_k^{(k)})$ is Cauchy.



Let $\tilde x$ denote the equivalence class containing $(y_k^{(k)})_{k=1}^{\infty}$. Then

$$\tilde d(\tilde y^{(n)}, \tilde x) = \lim_{k\to\infty} d(y_k^{(n)}, y_k^{(k)}) \le \lim_{k\to\infty}\lim_{i\to\infty}\Big(\underbrace{d(y_k^{(n)}, y_i^{(n)})}_{\le \frac{1}{\min(k,i)}} + d(y_i^{(n)}, y_i^{(k)}) + \underbrace{d(y_i^{(k)}, y_k^{(k)})}_{\le \frac{1}{\min(k,i)}}\Big) \le$$

$$\le \lim_{k\to\infty}\Big(\frac{1}{k} + \underbrace{\tilde d(\tilde y^{(n)}, \tilde y^{(k)})}_{\le \frac{1}{\min(n,k)}} + \frac{1}{k}\Big) = \frac{1}{n} \to 0 \quad \text{as } n \to \infty.$$

This proves that $\tilde y^{(n)} \to \tilde x$ in $\tilde X$; and since $\tilde y^{(n)}$ is a subseq. of the Cauchy seq. $(\tilde x^{(n)})$, it follows that $\tilde x^{(n)} \to \tilde x$.