APPM5440 — Applied Analysis: Section exam 3 — Solutions

17:15 – 18:30, Dec. 4, 2012. Closed books.

Problem 1: (28p) No motivation requireds — please just write the answers.

(a) Let X be a set, and let $\mathcal{T}_1$ and $\mathcal{T}_2$ be two topologies on X. Suppose that $\mathcal{T}_1$ is weaker than $\mathcal{T}_2$ (so that $\mathcal{T}_1 \subseteq \mathcal{T}_2$). Mark the true statements in the table below:

	Check if true:
If K is compact in $(X, \mathcal{T}_1)$, then K is compact in $(X, \mathcal{T}_2)$.	false
Suppose $f : \mathbb{R} \to (X, \mathcal{T}_1)$ is continuous, then $f : \mathbb{R} \to (X, \mathcal{T}_2)$ is continuous.	false
If $x_n \to x$ in $(X, \mathcal{T}_1)$, then $x_n \to x$ in $(X, \mathcal{T}_2)$.	false

(b) For which values of p is the Banach space ℓ^p reflexive? $1 < p < \infty$

(c) State the open mapping theorem:

(d) Set $X = \ell^2$ and $S = \{x \in \ell^2 : \ ||x|| = 1\}$. What is the closure of S in the weak topology?

The set $\{x \in \ell^2 : \ ||x|| \leq 1\}$

Problem 2: (28p) In this problem, you are given four pairs of linear spaces X and Y, and you are given some information about a map $T \in \mathcal{B}(X, Y)$.

(a) $X = Y = \ell^2$. $T(x_1, x_2, x_3, \dots) = (\frac{1}{1}x_1, \frac{1}{2}x_2, \frac{1}{3}x_3, \dots)$.
(b) $X = \ell^2$. Y is a normed linear space. We know that $||Tx|| = 2||x||$ for every $x \in X$.
(c) $X = Y = \ell^2$. We know that $||Tx|| \geq 2||x||$ for every $x \in X$.
(d) $X = \ell^2$. $Y = \mathbb{R}^n$. We know that $||Tx|| \leq 2||x||$ for every $x \in X$.

Fill out the table below. Write "yes" if the statement is necessarily true. Write "no" if it is necessarily false. Leave blank if there is not enough information. (No motivations required.)

	T is necessarily compact	ran(T) is necessarily closed
(a)	yes	no
(b)	no	yes
(c)	no	yes
(d)	yes	yes

(a) This was an exercise in class.

(b) Note that T is 1-to-1 and preserves the topology. As a consequence, ran(T) is homeomorphic to X. Since X is complete, ran(T) must be closed. Moreover, T is not compact — to see this, consider the sequence $(e_n)_{n=1}^{\infty}$ of the canonical basis vectors. The sequence is bounded, but $||Te_n - Te_m|| = 2||e_n - e_m|| = 2\sqrt{2}$ whenever $m \neq n$ so (Te_n) does not have a convergent subsequence.

(c) Since T is coercive, its range is necessarily closed. To see that T is not compact, use the same sequence as in the comments for (b).

(d) Recall that any finite dimensional linear space is closed, so ran(T) must be closed. That T is compact follows from the fact that any bounded set in a finite dimensional space is totally bounded.

Problem 3: (28p) In this problem, you are given four maps φ from different normed linear spaces X to $\mathbb{R}$. State which ones are elements of X^*. Motivate each answer briefly (in the space provided whenever possible).

(a) $I = [0, 1]$. $X = C(I)$ with standard norm. $\varphi(f) = \max_{x \in I} f(x)$.

Not in X^* since φ is not linear. For instance, consider $f_1 = x$ and $f_2 = 1 - x$. Then $\varphi(f_1) = 1$, $\varphi(f_2) = 1$, $\varphi(f_1 + f_2) = 1$.

(b) X is the space of polynomials of degree at most 2 and $||f|| = \sup_{x \in [0,1]} |f(x)|$. $\varphi(f) = f'(0)$.

In X^*. It is easy to prove that φ is linear and on a finite dimensional space, any linear map is continuous.

(c) $I = [0, 1]$. $X = C^1(I)$ with norm $||f|| = \sup_{x \in I} |f(x)|$. $\varphi(f) = f'(0)$.

Not in X^* since φ is not continuous. For instance, set $f_n(x) = \frac{1}{n} \sin(nx)$. Then

$$||\varphi|| \geq \lim_{n \to \infty} \frac{|\varphi(f_n)|}{||f_n||} = \lim_{n \to \infty} \frac{1}{1/n} = \infty.$$

(d) $I = [0, 1]$. $X = C^1(I)$ with standard norm. $\varphi(f) = \sum_{n=1}^{\infty} \frac{(-1)^n f(1/n)}{n}$.

In X^*. To see this, observe first that if $f \in X$, then

$$f(x) = f(0) + x\,\beta(x),$$

where for any $x \in I$, we have $\beta(x) = f'(t)$ for some $t \in [0, x]$. It follows that $|\beta(x)| \leq ||f'||_{\mathrm{u}} \leq ||f||$. Now

$$|\varphi(f)| = \left| \sum_{n=1}^{\infty} \frac{(-1)^n f(0)}{n} + \sum_{n=1}^{\infty} \frac{(-1)^n (1/n) \beta(1/n)}{n} \right| \leq |f(0)| \left| \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \right| + ||\beta||_{\mathrm{u}} \sum_{n=1}^{\infty} \frac{1}{n^2} \leq C\,||f||.$$

Note: The given functional is NOT continuous on $C(I)$. If your proof of continuity uses only that $|f(x)| \leq ||f||$, then it is wrong. Note that you can construct a sequence of continuous functions (f_n) such that $f_n(1/j) = (-1)^j$ for $j \in \{1, 2, \dots, n\}$, $f_n(1/j) = 0$ for $j > n$, and $||f_n||_{\mathrm{u}} = 1$. Then $\varphi(f_n) = \sum_{j=1}^{n} 1/j \to \infty$.

Problem 4: (16p) Let X denote a finite set, and let $\mathcal{T}$ be a metrizable topology on X. Prove that $\mathcal{T}$ is the discrete topology on X. (Write your solution in the space below, or on a separate sheet.)

See class notes.