APPM5440 — Applied Analysis: Section exam 2

17:15 – 18:30, Oct. 30, 2012. Closed books.

Please motivate all answers unless the problem explicitly states otherwise.
You may want to do Problem 5 last (it is only 20 points, and could be a lot of work).

Problem 1: (15p) State and prove the contraction mapping theorem.

Problem 2: (15p) State the Grönwall inequality.

Problem 3: (20p) Define for $n = 1, 2, 3, \dots$ the function $f_n : \mathbb{R} \to \mathbb{R}$ via

$$f_n(x) = e^{-n(x-n)^2}.$$

Let N be a fixed positive integer. In the table below, mark each box corresponding with a true statement with the letter "T". No motivations required.

	Ω is equicont. for every $x \in I$	Ω is uniformly equicont. on I	Ω is closed in $C(I)$	Ω is pre-compact in $C(I)$
$\Omega = \{f_n\}_{n=1}^{N}$ and $I = \mathbb{R}$				
$\Omega = \{f_n\}_{n=1}^{\infty}$ and $I = \mathbb{R}$				
$\Omega = \{f_n\}_{n=1}^{N}$ and $I = [-N, N]$				
$\Omega = \{f_n\}_{n=1}^{\infty}$ and $I = [-N, N]$				

Problem 4: (30p) Set $I = [0, 1]$.

(a) Let $(f_n)_{n=1}^{\infty}$ be a sequence of functions in $C(I)$ such that $\text{Lip}(f_n) \leq 1$. Prove that if $(f_n)_{n=1}^{\infty}$ converges uniformly to a function f, then $\text{Lip}(f) \leq 1$.

(b) Let $(f_n)_{n=1}^{\infty}$ be a sequence of functions in $C(I)$ such that $\text{Lip}(f_n) \leq 1$. Does (f_n) necessarily have a convergent subsequence? Please offer a proof or a counter-example.

(c) Set $\Omega = \{f \in C(I) : \text{Lip}(f) \leq 1 \text{ and } f(0) = 0\}$ Is the set Ω closed? Compact? Pre-compact?

(d) Is the set $\Omega = \{f \in C(I) : ||f|| \leq 1 \text{ and } \text{Lip}(f) \leq 1\}$ dense in the unit ball of $C(I)$?

Problem 5: (20p) Let $f = f(x, y)$ be a continuous bounded real-valued function on $\mathbb{R}^2$, and let $g = g(x)$ be a continuous real-valued function on $\mathbb{R}$ such that $||g||_{\text{u}} \leq 1$. Now consider for a positive number δ the equation

$$\begin{cases} u_1(x) = \displaystyle\int_0^{\delta} f(x, y)\,\big(u_2(y)\big)^2\,dy + g(x), \\ u_2(x) = \dfrac{1}{3} u_1(x) + \dfrac{1}{3}\big(u_2(x)\big)^2. \end{cases} \tag{1}$$

Show that for δ small enough, the equation (1) is guaranteed to have a unique solution pair (u_1, u_2) of continuous functions on $[0, \delta]$ such that $||u_2||_{\text{u}} \leq 1$. What can you say about $||u_1||_{\text{u}}$?