APPM5440 — Applied Analysis: Section exam 1

17:15 – 18:30, Sep. 25, 2012. Closed books.

Please motivate all answers unless the problem explicitly states otherwise.

Problem 1: (24 points) The following questions are worth 8 points each.

(a) Specify which of the following could potentially be the set C of cluster points of a sequence $(x_n)_{n=1}^{\infty}$ of real numbers. Any negative answer needs a brief motivation.
 (1) $C = [0, 1]$.
 (2) $C = (0, 1)$.
 (3) $C = [0, \infty)$.
 (4) $C = \mathbb{Q}$ (the set of rational numbers).
 (Recall that given a sequence (x_n), its set of cluster points is defined as the set of limit points of sub-sequences of (x_n).)

(b) Let (X, d) be a metric space. State the definition of the completion of (X, d).

(c) Which of the following statements are true (no motivations required):
 (1) If $(x_n)_{n=1}^{\infty}$ is a sequence of real numbers, then $\limsup_{n\to\infty} x_n$ exists.
 (2) If (X, d) is a compact metric space, and $(x_n)_{n=1}^{\infty}$ is a sequence in X with the property that every convergent subsequence has the same limit x, then $x_n \to x$.
 (3) Every compact subset of a metric space is necessarily closed.
 (4) If (X, d) is a compact metric space and $f : X \to (0, 1)$ is continuous, then the function $g(x) = 1/(1 - f(x))$ is bounded on X.
 (5) Let X be a normed linear space, and the B denote the unit ball around the origin. Then B is necessarily totally bounded.

Problem 2: (24 points) Suppose that (X_1, d_1), (X_2, d_2), and (X_3, d_3) are metric spaces, and that $f : X_1 \to X_2$ and $g : X_2 \to X_3$ are continuous. Prove that the composition $h = g \circ f$ defined by

$$h : X_1 \to X_3 : x \mapsto g(f(x))$$

is continuous. State explicitly which definition of continuity you use in your proof.

Problem 3: (24 points) Set $I = [-1, 1]$ and let X denote the set of real valued continuous functions on I. For $f \in X$, define the norm

$$||f|| = \int_{-1}^{1} |f(x)|\,dx.$$

Show that X is not a Banach space with respect to this norm.

Problem 4: (28 points) Let X denote the set of sequences of real numbers $x = (x_1, x_2, x_3, \dots)$ such that $\sum_{n=1}^{\infty} x_n^2 < \infty$, and define for $x \in X$ the norm $||x|| = \left(\sum_{n=1}^{\infty} x_n^2\right)^{1/2}$. Consider the following four subsets of X:

- Let d be a positive integer d and set $A = \{x = (x_1, x_2, \dots, x_d, 0, 0, \dots) : \sum_{n=1}^{d} x_n^2 \leq 1\}$.
- $B = \{x = (x_1, x_2, x_3, \dots) : \sum_{n=1}^{\infty} n^2 x_n^2 \leq 1\}$.
- $C = \{x = (x_1, x_2, x_3, \dots) : \sum_{n=1}^{\infty} x_n^2 \leq 1\}$.
- $D = \{x = (x_1, x_2, x_3, \dots) : \sum_{n=1}^{\infty} |x_n| = 1\}$.

Which of the sets A, B, C, and D are compact?