**Homework set 11 — APPM5440 — Fall 2012**

From the textbook: 5.11, 5.12, 5.13, 5.15a, (5.16), 5.17.

**Problem 1:** Let $X$ be a Banach space, and let $A, B \in \mathcal{B}(X)$ be two operators such that $AB = BA$. Prove that $e^{A+B} = e^{A}\,e^{B}$.

**Problem 2:** For $n = 1, 2, 3, \dots$, we define the operator $T_n$ on $X = l^2(\mathbb{N})$ by

$$T_n(x_1, x_2, x_3, \dots) = \frac{1}{\sqrt{n}}(x_1, x_2, \dots, x_n, 0, 0, \dots).$$

Prove that $T_n \in \mathcal{B}(X)$. Does $T_n$ converge to anything in norm? Strongly?

**Problem 3:** Let $X$ be an infinite dimensional Banach space and let $T \in \mathcal{B}(X)$ be a compact operator such that $\ker(T) = \{0\}$. Prove that $\operatorname{ran}(T)$ is not closed.

**Problem 4:** (Lax equivalence) Let $X$ and $Y$ be Banach spaces, let $A \in \mathcal{B}(X, Y)$ be an operator with a continuous inverse, let $f \in Y$, and consider the equation

$$A\,u = f.$$

Now suppose that we have "some mechanism" for approximating the equation to any given precision. In other words, given $\varepsilon > 0$, we can construct $A_\varepsilon$ that approximates $A$, and $f_\varepsilon$ that approximates $f$, and such that the equation

$$A_\varepsilon\,u_\varepsilon = f_\varepsilon$$

can be solved. (Typically, $A_\varepsilon$ is a finite dimensional operator, so that the approximate equation can be solved by solving a finite system of linear algebraic equations.) We say that

- The approximation is *consistent* if $A_\varepsilon \to A$ strongly.
- The approximation is *stable* if there is an $M < \infty$ such that $||A_\varepsilon^{-1}|| \leq M$ for all $\varepsilon > 0$.
- The approximation is *convergent* if $u_\varepsilon \to u$ whenever $f_\varepsilon \to f$ (in norm).

Suppose that the approximation scheme is consistent. Prove that then:

$$\text{The scheme is convergent} \quad \Leftrightarrow \quad \text{The scheme is stable}$$

*Hint:* The solution is in the text book, but please try it yourself before looking!

*Note:* In practice, variations of this result are often used in the context of approximating partial differential equations via, e.g., finite elements or finite differences. In this case, the operator is not bounded — this assumption can be done away with.