Comments on homework set 9 — APPM5440 — Fall 2012

Problem 4.1: Let K be a compact set and fix $x \notin K$. To prove that $K^{\rm c}$ is open (which is to say that K is closed) we need to construct an open set G such that $x \in G \subseteq K^{\rm c}$.

For every $y \in K$, pick disjoint open sets $G_y \ni x$ and $H_y \ni y$. Then $\{H_y\}_{y\in K}$ forms an open cover of K. Since K is compact, there is a finite subcover $\{H_{y_j}\}_{j=1}^{J}$. Now set

$$G = \bigcap_{j=1}^{J} G_j.$$

Clearly G is open (since its a finite intersection of open sets) and $x \in G$. Since G is also disjoint from every set H_{y_j} in the finite open cover of K, it follows that $G \cap K = \emptyset$ (which is to say $G \subseteq K^{\rm c}$).

Problem 4.2: For the cantor set C we have $\bar{C} = \partial C = C$ and $C^{\circ} = \emptyset$.

Problem 4.3: Just apply the definitions.

Problem 4.5a: The connected subspaces of $\mathbb{R}$ are the intervals.

Proof that any interval is connected: Let I be an interval, and suppose that $I = G_1 \cup G_2$ where G_1 and G_2 are both open, and $G_1 \cap G_2 = \emptyset$.

Pick a point $t \in I$. The point belongs to either G_1 or G_2. Say $t \in G_1$. Our claim is that then $G_2 \cap (t, \infty)$ must be empty. Suppose not, then set $s = \inf G_2 \cap (t,\infty)$. Since G_1 is open, it is not possible that $s \in G_1$ (if $s \in G_1$, there would be $\varepsilon > 0$ such that $B_\varepsilon(t) \subset G_1$ and then $\inf G_2 \cap (t,\infty) \geq s + \varepsilon$). Since G_2 is open, it not possible that $s \in G_2$ (if $s \in G_2$, there would be $\varepsilon > 0$ such that $B_\varepsilon(t) \subset G_2$ and then $\inf G_2 \cap (t,\infty) \leq s - \varepsilon$). Therefore $s \notin I$, which is impossible.

The proof that $G_2 \cap (-\infty, t)$ must be empty is analogous. It follows that G_2 must be empty.

Proof that any non-interval is not connected: Let I be a subset of $\mathbb{R}$ that is not an interval. Then there is a point $t \notin I$ such that the two sets

$$G_1 = (-\infty, t) \cap I, \qquad G_2 = (t, \infty) \cap I$$

are both non-empty. Both G_1 and G_2 are open in the subspace topology (by definition), they are non-intersecting, and $I = G_1 \cup G_2$.

Problem 1: Set $X = \mathbb{R}^2$ and $Y = \mathbb{R}$, and define $f : X \to Y$ by setting $f([x_1, x_2]) = x_1$. Prove that f is continuous. Prove that f is open. Prove that f does not necessarily map closed sets to closed sets.

Hint: That f is continuous is very easy to prove.

Let G be an open set in X. Pick $t \in f(G)$. There is some $x \in G$ such that $t = f(x)$. Since G is open, there is an $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq G$. But then $(t - \varepsilon,\, t + \varepsilon) = f(B_\varepsilon) \subseteq f(G)$ so $f(G)$ is open.

Consider the set $F = \{(x_1, x_2) \in \mathbb{R}^2 : \ x_1 x_2 = 1\}$. Then F is closed. But $f(F) = (-\infty, 0) \cup (0, \infty)$ which is not closed.

Problem 2: Prove that the co-finite topology is first countable if and only if X is countable.

Hint: Read the definitions carefully — there is nothing tricky about this question.

Problem 3: Prove that the co-finite topology on $\mathbb{R}$ weaker than the standard topology.

Hint: Note that all you need to do is to demonstrate that any set that is open in the cofinite topology is also open in the standard topology.