Homework set 9 — APPM5440 — Fall 2012

From the textbook: 4.1, 4.2, 4.3, 4.5a, 4.6.

Problem 1: Set $X = \mathbb{R}^2$ and $Y = \mathbb{R}$, and define $f : X \to Y$ by setting $f([x_1, x_2]) = x_1$. Prove that f is continuous. Prove that f is open. Prove that f does not necessarily map closed sets to closed sets.

Problem 2: Prove that the co-finite topology is first countable if and only if X is countable.

Problem 3: Prove that the co-finite topology on $\mathbb{R}$ weaker than the standard topology.

The last two problems are entirely optional.

Problem 4: The Hausdorff property is only one of many so called "separability" conditions on topological spaces. As an example, we say that a topological space X is T_j, for $j = 0, 1, 2, 3, 4$ if:

- T_0**:** Given $x, y \in X$, there either exists an open set containing x but not y, or vice versa.
- T_1**:** Given $x, y \in X$, there exists an open set that contains x but not y.
- T_2**:** Given $x, y \in X$, there exist disjoint open sets G, H such that $x \in G$, $y \in H$. (Note that T_2 is the same as Hausdorff.)
- T_3**:** X is T_1, and: Given any closed set A, and any point $x \in A^{\mathrm{c}}$, there exist disjoint open sets G, H such that $x \in G$, $A \subseteq H$.
- T_4**:** Given any two closed disjoint sets A and B, there exists disjoint open set G, and H such that $A \subseteq G$, $B \subseteq H$.

Prove that if $i < j$, then any T_j space is T_i. Prove that the co-finite topology is T_1 but not T_2. Prove that a topological space is T_1 if and only if the set $\{x\}$ is closed for every $x \in X$.

Problem 5: Consider the set $X = \mathbb{R}$. Let $\mathcal{S}$ denote the collection of sets of the form $(-\infty, a]$ or (a, ∞) for $a \in \mathbb{R}$.

(a) Let $\mathcal{B}$ denote the collection of sets obtained by taking finite intersections of sets in $\mathcal{S}$. Prove that if $G \in \mathcal{B}$, then either G is empty, or $G = (a, b]$ for some a and b such that $-\infty < a < b < \infty\}$.
(b) Let $\mathcal{T}$ denote the topology generated by the base $\mathcal{B}$. Prove that all sets in $\mathcal{B}$ are both open and closed in $\mathcal{T}$.
(c) Prove that $\mathcal{T}$ is first countable but not second countable. Hint: For any $x \in X$, any neighborhood base at x contains at least one set whose supremum is x.
(d) Prove that $\mathbb{Q}$ is dense in $\mathcal{T}$. (This proves that $(X, \mathcal{T})$ is separable but not second countable.)
(e) Prove that $(X, \mathcal{T})$ is not metrizable.