**Solutions to homework set 7 — APPM5440 — Fall 2012**

**Problem 3.1:** Suppose $T(x) = x$. Then $\pi/2 - \arctan(x) = 0$ which clearly is impossible.

Set

$$\alpha \sup_{x \neq y} \frac{d(T(x), T(y))}{d(x, y)}.$$

The CMT holds only if $\alpha \in (0, 1)$. In other words, there must be a single $\alpha$ such that the relation $d(T(x), T(y)) \leq \alpha\, d(x, y)$ holds for every pair $\{x, y\}$. In the present case $\alpha = 1$.

**Problem 3.4:** For any $n$, we have

$$d(x_n, x_0) \leq d(x_n, x_{n-1}) + d(x_{n-1}, x_{n-2}) + \cdots + d(x_1, x_0).$$

Take the limit as $n \to \infty$ to get

$$d(x, x_0) \leq \sum_{n=1}^{\infty} d(x_n, x_{n-1}).$$

Now

$$d(x_n, x_{n-1}) \leq c\, d(x_{n-1}, x_{n-2}) \leq c^2\, d(x_{n-2}, x_{n-3}) \leq \cdots \leq c^{n-1}\, d(x_1, x_0).$$

Combine the two inequalities to complete the proof.

**Problem 3.5:** We use the matrix norm

$$||S|| = \max_{i=1,\,2,\,3,\,\dots,\,n} \sum_{j=1}^{n} |s_{i,j}|.$$

With $|x| = \max_{i=1,\,2,\,3,\,\dots,\,n} |x_i|$, we then have

$$|Sx| \leq ||S||\, |x|.$$

*Jacobi:* The iteration map is

$$T(x) = D^{-1}(L + U)\, x + D^{-1}\, b.$$

We show that $T$ is a contraction:

$$|T(x) - T(y)| = |D^{-1}(L + U)\, (x - y)| \leq ||D^{-1}(L + U)||\, |x - y|.$$

Now

$$||D^{-1}(L + U)|| = \max_{i=1,\,2,\,3,\,\dots,\,n} \sum_{j \neq i}^{n} \left| \frac{a_{i,j}}{a_{i,i}} \right| < 1$$

by the assumption that $A$ is strictly row diagonally dominant. The iteration $x_{n+1} = T(x_n)$ converges by the CMT to a point $x$ such that $T(x) = x$, which is to say $x = D^{-1}(L+U)x + D^{-1}b$. Multiply by $D$ to get $Dx = Lx + Ux + b$, or $(D - L - U)x = b$.

*Gauss-Seidel:* The iteration map is

$$T(x) = (D - L)^{-1} U\, x + D^{-1}\, b.$$

Set $B = (D - L)^{-1} U$, and set

$$\alpha = \max_{i=1,\,2,\,3,\,\dots,\,n} \sum_{j \neq i}^{n} \left| \frac{a_{i,j}}{a_{i,i}} \right|.$$

By assumption (strict row diagonal dominance), we have $\alpha < 1$. Fix $x$ and set $y = Bx$. We will show that $|y| \leq \alpha|x|$. First consider the element $y_1$, we find

$$|y_1| = \left|\sum_{j>1} \frac{a_{1,j}}{a_{11}} x_j\right| \leq \sum_{j>1} \left|\frac{a_{1,j}}{a_{1,1}}\right| |x| \leq \alpha|x|.$$

Next consider $y_2$:

$$|y_2| = \left|\sum_{j<2} \frac{a_{2,j}}{a_{22}} y_j + \sum_{j>2} \frac{a_{2,j}}{a_{22}} x_j\right| \leq \sum_{j<2} \left|\frac{a_{2,j}}{a_{22}}\right| |y_j| + \sum_{j>2} \left|\frac{a_{2,j}}{a_{22}}\right| |x_j| \leq \sum_{j<2} \left|\frac{a_{2,j}}{a_{22}}\right| |x_j| + \sum_{j>2} \left|\frac{a_{2,j}}{a_{22}}\right| |x_j| \leq \alpha|x|.$$

In the second inequality, we used that $|y_1| \leq |x|$. Next,

$$|y_3| = \left|\sum_{j<3} \frac{a_{3,j}}{a_{33}} y_j + \sum_{j>3} \frac{a_{3,j}}{a_{33}} x_j\right| \leq \sum_{j<3} \left|\frac{a_{3,j}}{a_{33}}\right| |y_j| + \sum_{j>3} \left|\frac{a_{3,j}}{a_{33}}\right| |x_j| \leq \sum_{j<3} \left|\frac{a_{3,j}}{a_{33}}\right| |x_j| + \sum_{j>3} \left|\frac{a_{3,j}}{a_{33}}\right| |x_j| \leq \alpha|x|.$$

In the second inequality, we used that $|y_1| \leq |x|$ and that $|y_2| \leq |x|$.

Continuing the process outlined through all $n$ steps, we find

$$|y| \leq \alpha\,|x|.$$

Now use the CMT to assert convergence of the iteration.

The proof that the limit point satisfies $Ax = b$ goes exactly like in the Jacobi case.