APPM 5440, Fall 2012.

Notes on Chapter 5 – Banach Spaces

Most important topics: You are expected to know these definitions and results well.

- Definition of a Banach space. The spaces $\ell^p(\mathbb{N})$, $C_\mathrm{b}(I)$, $C^k_\mathrm{b}(I)$.
- The space $\mathcal{B}(X, Y)$. The operator norm (equations (5.2) and (5.3) are important). Strong convergence. Norm convergence implies strong convergence.
- For a linear operator: Continuity $\Leftrightarrow$ Boundedness. (Thm. 5.18)
- Equivalent norms.
- Properties of the kernel and the range of a linear operator. The kernel of a continuous operator is (topologically) closed.
- Simplifications in finite-dimensional spaces (all linear operators are bounded, all norm topologies are equivalent, *etc*).
- Theorem 5.37 ($||ST|| \leq ||S||\,||T||$).
- Definition of a compact operator. Prop. 5.43.
- Definition of the (topological) dual X^* of a normed linear space X.
- In a Banach space X, the definition of weak convergence is that $x_n \rightharpoonup x$ if and only if $\varphi(x_n) \to \varphi(x)\ \forall \varphi \in X^*$. Norm convergence implies weak convergence.

Important topics: Study these concepts once you master the core concepts above. They are no less important, but you may find them more challenging. (These topics are not off-bounds for exam questions.)

- Isomorphisms between Banach spaces.
- The open mapping theorem (statement only, not the proof).
- Coercive operators have closed range (Prop. 5.30).
- The Hahn-Banach theorem. The linear functionals separate points in X. The elements of X separate points in X^* (so that the weak-$*$ topology on X^* is Hausdorff).
- Definition of the exponential of an operator — proof that the sum converges in norm.
- The unit ball in a reflexive Banach space is compact in the weak topology.

"Extra credit" topics: The following topics are included primarily as orientation. There *may* be exam problems that touch upon these topics, but you can do well on the exam as long as you know the core topics listed above.

- Extension of a bounded linear operator defined on a dense set (Thm 5.19).
- Differential equations in Banach spaces. Semi-groups.
- Lax equivalence.
- Weak-$*$ convergence in X^*. Alaoglu's theorem. Isometric embedding of X into X^{**}.

What type of questions to expect:
On the midterm and on the final, some questions will be of the type below. (There may of course be other types of questions as well, but these are archetypical ones.)

- Consider the Banach space X defined by .... Consider the vectors $x_n \in X$ defined by .... Does the sequence (x_n) converge weakly? If so, to what? Does x_n converge in norm? If so, to what?
- Consider the Banach spaces X and Y defined by .... Consider the sequence of operators $T_n \in \mathcal{B}(X, Y)$ defined by .... Prove that (T_n) converges strongly but not in norm. What is the strong limit?
- Consider the Banach space X defined by ..., and the operator $T \in \mathcal{B}(X)$ defined by .... Prove that T is compact.
- Consider the Banach space X defined by .... Give an example of a sequence in X that is weakly convergent, but not norm convergent.
- Consider the Banach space X defined by .... The vectors $\{e_n\}_{n=1}^{\infty}$ where $e_n = \cdots$ form a basis for X. Consider the operator $T \in \mathcal{B}(X)$ defined by .... What is the matrix of T in the basis (e_n)?
- Consider the Banach space X defined by .... Consider the operator $T \in \mathcal{B}(X)$ defined by .... Prove that T maps weakly convergent sequences to strongly convergent sequences.
- Consider the Banach space X defined by .... Consider the operator $T \in \mathcal{B}(X)$ defined by .... Prove that the range of T is not topologically closed.