THE LINEAR SPACE $\mathcal{B}(X, Y)$

Let X & Y be NLS's.

Then $\mathcal{B}(X,Y)$ is the set of all bdd linear maps $X \to Y$.

We equip $\mathcal{B}(X,Y)$ with the norm

$$\|T\| = \sup_{x \neq 0} \frac{\|Tx\|_Y}{\|x\|_X} = \sup_{\|x\|_X = 1} \|Tx\|_Y .$$

Then $\mathcal{B}(X,Y)$ is a NLS in its own right.

<u>Thm</u> Let X be a NLS & let Y be a Banach space. Then $\mathcal{B}(X,Y)$ is a Banach space.

<u>Proof</u> We need to prove completeness.

Let $(T_n)_{n=1}^{\infty}$ be a Cauchy seq.

* First we construct a putative limit point:

Given $x \in X$, the seq $(T_n x)_{n=1}^{\infty}$ is Cauchy since $\|T_n x - T_m x\| \leq \|T_n - T_m\| \, \|x\|$.

Since Y is complete, $(T_n x)$ has a limit point, define Tx as this limit point.

* Is T linear?

Fix $\alpha, \beta \in \mathbb{R}$ and $x, y \in X$. Then

$$T(\alpha x + \beta y) = \lim_{n \to \infty} T_n(\alpha x + \beta y) = \lim_{n \to \infty} [\alpha T_n x + \beta T_n y] = \alpha Tx + \beta Ty$$ Yes!

* Is T bdd?

Since (T_n) is Cauchy, we have $C = \lim_{n \to \infty} \|T_n\| < \infty$.

Then $\|Tx\| = \lim_n \|T_n x\| \leq \limsup_n \|T_n\| \, \|x\| = C \|x\|$

Yes.

* It remains to prove that $\|T_n - T\| \to 0$ as $n \to \infty$.
Fix $\varepsilon > 0$. Since (T_n) is Cauchy, $\exists N$ s.t. $m,n \geq N \Rightarrow \|T_m - T_n\| < \varepsilon$.

Fix x s.t. $\|x\| = 1$. Then if $m,n \geq N$,

$$\|(T_m - T_n)x\| \leq \|T_m - T_n\| \, \|x\| < \varepsilon. \qquad (1)$$

Take limit as $m \to \infty$ in (1):

$$\|(T - T_n)x\| = \lim_{m\to\infty} \|T_m x - T_n x\| \leq \varepsilon \qquad (2)$$

Take sup over x in (2):

$$\|T - T_n\| = \sup_{\|x\|=1} \|(T - T_n)x\| \leq \varepsilon.$$

<u>Def^n</u> A linear space L is called an <u>ALGEBRA</u> if there is a multiplication operator $L \times L \to L$ that satisfies

(1) $(xy)z = x(yz) \qquad \forall\, x,y,z \in L$

(2) $x(y+z) = xy + xz \qquad \forall\, x,y,z \in L$

(3) $\alpha(xy) = (\alpha x)y = x(\alpha y) \quad \forall\, \alpha \in \mathbb{R} \;\; x,y \in L$

A <u>Banach algebra</u> is a Banach space that is also an algebra with a multiplication satisfying

(4) There is an element $e \in L$ s.t. $xe = ex = x \quad \forall x \in L$

(5) $\|xy\| \leq \|x\| \, \|y\| \qquad \forall x,y \in L$

Examples of algebras:

* $\mathbb{C}$ and $\mathbb{R}$ themselves
* $C_b(X)$ for any topological space X.
  $C_b(X)$ is in fact a Banach algebra.
* Let Ω be a domain in $\mathbb{C}$.
  Then $A(\Omega)$ = space of analytic functions on Ω
  is a Banach algebra (with the uniform norm).
* If X is an NLS, then $\mathcal{B}(X) = \mathcal{B}(X,X)$ is an algebra.
  If X is a Banach space, then $\mathcal{B}(X)$ is a Banach algebra.

<u>Note</u> $\mathcal{B}(X)$ is almost never commutative!

<u>Def^n</u> Suppose that $T_n, T \in \mathcal{B}(X,X)$.
We say that $T_n \to T$ <u>in norm</u> if $\|T_n - T\| \to 0$ as $n \to \infty$.
We say that $T_n \to T$ <u>strongly</u> if $\|T_n x - Tx\| \to 0$ as $n \to \infty$ $\forall x$.

<u>Note:</u> Norm convergence implies strong convergence but the converse is not necessarily true.

<u>Example</u> $X = \ell^2$ (with standard norm)

$P_n : x \mapsto (x_1, x_2, \ldots, x_n, 0, 0, \ldots)$

$P_n \to I$ strongly since for any x we have $P_n x \to x$.

$P_n \not\to I$ in norm since $\|P_n - I\| = 1$ for all n.

Example Let $\mathfrak{X}$ be a Banach space and suppose that $A \in \mathcal{B}(\mathfrak{X})$.

Set $T_n = \sum_{j=0}^{n} \frac{1}{j!} A^j$, then (T_n) is Cauchy since

$$\|T_n - T_m\| = \Big\|\sum_{j=m+1}^{n} \frac{1}{j!} A^j\Big\| \le \sum_{j=m+1}^{n} \frac{\|A\|^j}{j!} \text{''} \to 0 \quad \text{as } m,n \to \infty \text{''}$$

Thus T_n converges in norm to an element $\exp(A)$

$$\exp(A) = \sum_{j=0}^{\infty} \frac{1}{j!} A^j. \qquad \|\exp(A)\| \le e^{\|A\|}.$$

If $\mathfrak{X} = \mathbb{R}^n$ & $A = V \overset{\text{diagonal}}{D} V^{-1}$, then $\exp(A) = V \begin{bmatrix} e^{\lambda_1} & 0 & & \\ 0 & e^{\lambda_2} & & \\ & & \ddots & \\ & & & e^{\lambda_n} \end{bmatrix} V^{-1}$

If A and B commute, then $\exp(A+B) = \exp(A)\exp(B)$

Now consider the ODE
$$\begin{cases} \frac{d}{dt} x(t) = A\, x(t) \\ x(0) = x_0 \end{cases}$$

where $\frac{d}{dt}$ is defined as the uniform ~~norm~~ limit $\lim_{h\to 0} \frac{x(t+h) - x(t)}{h}$.

The solution is $x(t) = \exp(At)\, x_0$

Proof:
$$\Big\| \frac{1}{h}(x(t+h) - x(t)) - Ax(t)\Big\| = \Big\| \frac{1}{h}\left(e^{A(t+h)}x_0 - e^{At}x_0\right) - Ae^{At}x_0\Big\| =$$
$$= \Big\| e^{At}\left(\frac{1}{h}\left(e^{Ah} - I\right) - A\right)x_0\Big\| = \Big\| e^{At}\frac{1}{h}\left(\sum_{n=1}^{\infty}\frac{(Ah)^n}{n!} - A\right)x_0\Big\| =$$
$$= \Big\| e^{At}\left(\sum_{n=2}^{\infty}\frac{A^n h^{n-1}}{n!}\right)x_0\Big\| \le h\, \|e^{At}\| \underbrace{\left(\sum_{n=2}^{\infty}\frac{\|A\|^n}{n!}\right)}_{\le e^{\|A\|}} \|x_0\| \to 0 \text{ as } h \to 0.$$