Applied Analysis (APPM 5440): Section exam 3
8:30am – 9:50am, Nov. 30, 2009. Closed books.

Problem 1: (24p) With X a Banach space, which statements are necessarily true (please motivate):

(a) If $S, T \in \mathcal{B}(X)$ and T is compact, then ST is compact.

(b) If $S, T \in \mathcal{B}(X)$ and T is compact, then TS is compact.

(c) Suppose that for $n = 1, 2, 3, \dots$, we know that $T_n \in \mathcal{B}(X)$ has finite dimensional range, and that there exists an operator $T \in \mathcal{B}(X)$ such that T_n converges strongly to T. Then T is compact.

Problem 2: (28p) Set $X = \ell^3$. Define the operator $T \in \mathcal{B}(X)$ via

$$T\,(x_1,\, x_2,\, x_3,\, \dots) = (\tfrac{1}{1}x_1,\, \tfrac{1}{2}x_2,\, \tfrac{1}{3}x_3,\, \dots).$$

Which of the following statements are necessarily true? Motivate your answers.

(a) $\mathrm{Ran}(T)$ is a linear subspace.

(b) $\mathrm{Ker}(T)$ is a linear subspace.

(c) $\mathrm{Ran}(T)$ is topologically closed.

(d) $\mathrm{Ker}(T)$ is topologically closed.

Problem 3: (24p) Set $X = \ell^\infty$. Define for any positive integer n a linear map φ_n from X to $\mathbb{R}$ via

$$\varphi_n(x) = \frac{1}{n}\sum_{j=1}^{n} x_j.$$

(a) Prove that φ_n is bounded and determine its norm.

(b) Does $(\varphi_n)_{n=1}^{\infty}$ converge in norm in X^*?

(c) Does $(\varphi_n)_{n=1}^{\infty}$ converge weakly in X^*?

For 6 points extra credit: Answer (a), (b), and (c), again, but now for the space $X = \ell^1$.

Problem 4: (24p) Let X be a topological space that satisfies the Hausdorff property. Let K be a compact subset of X.

(a) State the definition of the Hausdorff property.

(b) State the definition of a compact set in a general topological space.

(c) Prove that K is necessarily closed.