Homework set 8 — APPM5440 — Solutions

3.6: Set $X = C([-a, a])$ and define on X the operator

$$[Fu](x) = \frac{1}{\pi} \int_{-a}^{a} \frac{1}{1 + (x - y)^2}\, u(y)\, dy + 1.$$

Then the given equation can be formulated as a fixed point problem $u = F(u)$. We find that

$$||F(u) - F(v)|| = \sup_{x \in [-a,a]} \left| \frac{1}{\pi} \int_{-a}^{a} \frac{1}{1 + (x - y)^2}\, (u(y) - v(y))\, dy \right|$$

$$\leq \cdots \leq \frac{2 \arctan(a)}{\pi} ||u - v||$$

(you should be able to fill in the missing steps). The contraction mapping principle now proves that the equation has a unique solution in X.

To prove that the solution is positive, note that if $u(x) \geq 0$ for all x, then $[Tu](x) \geq 0$ for all x. Combine this fact with the fact that f is non-negative, and that

$$u = (I - T)^{-1} f = \sum_{n=0}^{\infty} T^n f.$$

For the case $a = \infty$, note that uniqueness cannot hold. If u is a solution, then so is any shift of u.

3.7: Set $X = \{u \in C(I) : u(0) = u(1) = 0\}$. Convolve given BVP with the Green's function $g(x, y)$ (defined by eqn (3.22)) and obtain

$$u + \lambda T\, u = h, \tag{1}$$

where T is the operator on X given by

$$[Tu](x) = \int_0^1 g(x, y)\, \sin(u(y))\, dy,$$

and

$$h(x) = \int_0^1 g(x, y)\, f(y)\, dy.$$

Note that $Tf \in X$, so (1) is an equation on X. Now

$$||Tu - Tv|| \leq \sup_x \int_0^1 |g(x, y)|\, |\sin(u(y)) - \sin(v(y))|\, dy.$$

Use that

$$|\sin(u(y)) - \sin(v(y))| \leq |u(y) - v(y)| \leq ||u - v||$$

to obtain

$$||Tu - Tv|| \leq \beta ||u - v||,$$

where

$$\beta = \sup_x \int_0^1 |g(x, y)|\, dy.$$

We see that if $\lambda < 1/\beta$, equation (1) has a unique solution in X (by the contraction mapping theorem).

Problem 4: Consider the integral equation

$$(*) \qquad u(x) = \pi^2 \sin(x) + \frac{3}{2}\int_0^{\cos(x)} |x-y|\, u(y)\, dy.$$

Prove that (*) has a unique solution in $C([0,1])$.

Solution: Set $X = C([0,1])$ and define the operator T on X by

$$[Tu](x) = \frac{3}{2}\int_0^{\cos(x)} |x-y|\, u(y)\, dy.$$

We find that

$$\begin{aligned} ||Tu - Tv|| &\leq \sup_x \frac{3}{2}\int_0^{\cos(x)} |x-y|\, |u(y)-v(y)|\, dy \\ &\leq \sup_x \frac{3}{2}\int_0^{1} |x-y|\, |u(y)-v(y)|\, dy \\ &\leq \sup_x \frac{3}{2}\int_0^{1} |x-y|\, dy\, ||u-v||. \end{aligned}$$

Now

$$\sup_x \int_0^1 |x-y|\, dy = \sup_x (\frac{1}{2}x^2 + \frac{1}{2}(1-x)^2) = \frac{1}{2},$$

so

$$||Tu - Tv|| \leq \frac{3}{4}||u-v||.$$

Since T is a contraction, the equation

$$(I - T)\, u = f$$

has a unique solution for every f. (In particular, for $f(x) = \pi^2 \sin(x)$.)

Problem 1: Let X be a set with infinitely many members. We define a collection $\mathcal{T}$ of subsets of X by saying that a set $\Omega \in \mathcal{T}$ if either $\Omega^{\rm c} = X \backslash \Omega$ is finite, or if Ω is the empty set. Verify that $\mathcal{T}$ is a topology on X. This topology is called the "co-finite" topology on X. Describe the closed sets.

Solution: The verification should be straight-forward. The closed sets are the finite sets, and the entire set.

Problem 2: Let X denote a finite set, and let $\mathcal{T}$ be a metrizable topology on X. Prove that $\mathcal{T}$ is the discrete topology on X.

Solution: Enumerate the points in X so that $X = \{x_n\}_{n=1}^N$. Set

$$\varepsilon_n = \min\{d(x_n, x_m) : \; m \neq n\}, \quad \text{for } n = 1, 2, \dots, N.$$

Since the minimum is taken over a finite set of positive numbers, $\varepsilon_n > 0$, and it follows that the set $B_{\varepsilon_n/2}(x_n) = \{x_n\}$, must be open. Since any union of open set must itself be open, it follows that all subsets of X are open.

Problem 3: Consider the set $X = \{a,\, b,\, c\}$, and the collection of subsets $\mathcal{T} = \{\emptyset,\, \{a\},\, \{a, b\},\, \{a, c\},\, \{a, b, c\}\}$. Is $\mathcal{T}$ a topology? Is $\mathcal{T}$ a metrizable topology?

Solution: Yes it is a topology (and union or intersection of the given sets is itself a member of the set.

No, it cannot be metrizable. If it were, then the argument given in Problem 2 would demonstrate that $\{b\}$ would have to be an open set, and it is not.