Homework set 8 — APPM5440 — Fall 2009

From the textbook: 3.6, 3.7.

On the next page, you'll find the 2005 midterm. Problem 4 on that midterm is part of this week's homework. (Note that the questions on topological spaces are outside the syllabus for the midterm this year.)

Problem 1: Let X be a set with infinitely many members. We define a collection $\mathcal{T}$ of subsets of X by saying that a set $\Omega \in \mathcal{T}$ if either $\Omega^{\mathrm{c}} = X \backslash \Omega$ is finite, or if Ω is the empty set. Verify that $\mathcal{T}$ is a topology on X. This topology is called the "co-finite" topology on X. Describe the closed sets.

Problem 2: Let X denote a finite set, and let $\mathcal{T}$ be a metrizable topology on X. Prove that $\mathcal{T}$ is the discrete topology on X.

Problem 3: Consider the set $X = \{a,\, b,\, c\}$, and the collection of subsets $\mathcal{T} = \{\emptyset,\, \{a\},\, \{a,b\},\, \{a,c\},\, \{a,b,c\}\}$. Is $\mathcal{T}$ a topology? Is $\mathcal{T}$ a metrizable topology?

Applied Analysis (APPM 5440): Midterm 2
5.00pm – 6.30pm, Oct 26, 2005. Closed books.

Problem 1: State the Arzelà-Ascoli theorem.

Problem 2: We describe two mathematical objects below. For each description, either provide an example of such an object, or explain why it does not exist. (Brief answers, please!)

(1) A collection of functions in $C(\mathbb{R})$ that is equicontinuous but not uniformly equicontinuous.
(2) A collection of functions in $C([0,1])$ that is equicontinuous but not uniformly equicontinuous.

Problem 3: Let $X = \{a, b, c, d\}$ be a set, and let

$$\mathcal{S} = \{\emptyset, X, \{b, d\}, \{a, c\}, \{d\}, \{a, b, c\}\}$$

be a collection of subsets of X.

(1) Prove that $\mathcal{S}$ is not a topology on X.
(2) Let $\mathcal{T}$ denote the smallest topology on X that contains $\mathcal{S}$ (in other words, $\mathcal{T}$ is the topology generated by the sub-base $\mathcal{S}$). List the sets that are contained in $\mathcal{T}$ but not in $\mathcal{S}$.

Problem 4: Consider the integral equation

$$(*) \qquad u(x) = \pi^2 \sin(x) + \frac{3}{2} \int_0^{\cos(x)} |x - y|\, u(y)\, dy.$$

Prove that (*) has a unique solution in $C([0,1])$.

Problem 5: Let X be a topological space, let Y be a Hausdorff space, and let f and g be continuous maps from X to Y. Is it necessarily the case that the set $\Omega = \{x \in X : f(x) = g(x)\}$ is closed? Justify your answer by either giving a proof or a counter example.