Homework set 11 — APPM5440, Fall 2009

From the textbook: 5.11, 5.12, 5.13, 5.15a, (5.16), 5.17.

Problem 1: For $n = 1, 2, 3, \dots$, we define the operator T_n on $X = l^2(\mathbb{N})$ by

$$T_n(x_1, x_2, x_3, \dots) = \frac{1}{\sqrt{n}}(x_1, x_2, \dots, x_n, 0, 0, \dots).$$

Prove that $T_n \in \mathcal{B}(X)$. Does T_n converge to anything in norm? Strongly?

Problem 2: Let X be an infinite dimensional Banach space and let $T \in \mathcal{B}(X)$ be a compact operator such that $\ker(T) = \{0\}$. Prove that $\mathrm{ran}(T)$ is not closed.

Problem 3: Let X be a finite dimensional space with a basis $\{e^{(j)}\}_{j=1}^{d}$. Define a putative norm on X by setting

$$||x|| = ||\sum_{j=1}^{d} x_j\, e^{(j)}|| = \sum_{j=1}^{d} |x_j|.$$

(a) Prove that $||\cdot||$ is in fact a norm on X.

(b) Prove that the set $K = \{x \in X : \ ||x|| = 1\}$ is compact (in the topology defined by $||\cdot||$).

(c) Let $|||\cdot|||$ denote a different norm on X. Set

$$f : \ X \to \mathbb{R} : \ x \mapsto |||x|||.$$

Prove that f is continuous.