Applied Analysis (APPM 5440): Final exam

1:30pm – 4:00pm, Dec. 14, 2009. Closed books.

Problem 1: (20p) Set $I = [0, 1]$. Prove that there is a continuous function u on I such that

$$u(x) - \frac{1}{5}\int_0^x \sin\big(u(t)^2\big)\,dt = \cos(x), \qquad x \in I.$$

Problem 2: (25p) Let X be a set.

(a) (6p) State the definitions of a metric on X and a topology on X.

(b) (4p) Given a metric d on X, define the topology $\mathcal{T}$ induced by d.

(c) (8p) Prove that the $\mathcal{T}$ that you defined in (b) satisfies the axioms of a topology.

(d) (7p) Set $Y = \mathbb{R}^2$, and define on $Y \times Y$ the function

$$b(x, y) = b([x_1, x_2], [y_1, y_2]) = \big(|x_1 - y_1|^{1/2} + |x_2 - y_2|^{1/2}\big)^2.$$

Is (Y, b) a metric space? Motivate.

Problem 3: (10p)

(a) (5p) State the Hahn-Banach theorem.

(b) (5p) Define what it means for a sequence $(x_n)_{n=1}^{\infty}$ in a Banach space X to converge weakly.

Problem 4: (25p) Let H be a Hilbert space with a closed convex subset M.

(a) (13p) Suppose that $x \in H$ and that $x \notin M$. Prove that there exists a unique $z \in M$ such that

$$||x - z|| = \inf_{y \in M} ||x - y||.$$

(b) (12p) Now consider the particular case of $H = L^2(I)$ where $I = [0, 1]$. The set H is equipped with the usual inner product $(f, g) = \int_0^1 \overline{f(t)}\, g(t)\, dt$. Let M denote the linear space of polynomials of degree two or less, and set $f(t) = t^3$. Evaluate

$$d = \mathrm{dist}(M, f) = \inf_{g \in M} ||f - g||.$$

Note: In grading part (b), the priority will be placed on whether you clearly explain the steps that you take. If you do, you will get close to full credit even if some numbers are incorrect, or, if your answer involves unevaluated constants.

Problem 5: (20p) Let (X, d) be a compact metric space. Let $C_{\mathrm{b}}(X)$ denote the set of all bounded real-valued continuous functions on X, equipped with the uniform norm,

$$||f||_{\mathrm{u}} = \sup_{x \in X} |f(x)|.$$

Prove that $C_{\mathrm{b}}(X)$ is complete.