Applied Analysis (APPM 5440): Midterm 3
5.30pm – 6.50pm, Dec. 4, 2006. Closed books.

Problem 1: No motivation required. 2p each:

(a) Let $(X, \mathcal{T})$ denote a topological space. Specify the axioms that $\mathcal{T}$ must satisfy.

(b) Let $(X, \mathcal{T})$ denote a topological space. Define what it means for $\mathcal{T}$ to be Hausdorff.

(c) Let $(X, \mathcal{T})$ denote a topological space, let $(x_n)_{n=1}^{\infty}$ denote a sequence in X, and let x denote an element of X. Define what it means for x_n to converge to x. ($\mathcal{T}$ is not necessarily metrizable.)

Problem 2: Consider the set $X = \{a,\, b,\, c\}$, and the collection of subsets $\mathcal{T} = \{\emptyset,\, \{a\},\, \{b, c\},\, \{a, b, c\}\}$. Is $\mathcal{T}$ a metrizable topology? List the compact subsets of X. Give an example of a function $f: X \rightarrow \mathbb{R}$ that is continuous, and one example of a function $g: X \rightarrow \mathbb{R}$ that is not. Justify your answers briefly. (6p)

Problem 3: Let X denote the set of all continuous functions on the interval $I = [-\pi,\, \pi]$. Equip X with the norm

$$||f|| = \int_{-\pi}^{\pi} |f(y)|\, dy.$$

Consider the operator $T \in \mathcal{B}(X)$ that is defined by

$$[Tf](x) = \int_0^{\pi} \sin(x)\, y^2\, f(y)\, dy.$$

Calculate the norm of T in $\mathcal{B}(X)$. (4p total: 2p for the correct answer α, and 1p each for the proofs that $\alpha \leq ||T||$ and that $\alpha \geq ||T||$.)

Problem 4: Let X be a Banach space with a compact subset K. Suppose that $(x_n)_{n=1}^{\infty}$ is a sequence of elements in K that converges weakly to some element $x \in K$. Is it necessarily the case that the sequence also converges in norm to x? Either prove that this is the case, or give a counter-example. (4p)

Problem 5: Consider the Banach space $X = l^2(\mathbb{N})$, and the operator $T \in \mathcal{B}(X)$ defined by

$$Tx = (\tfrac{1}{1}\, x_1,\, \tfrac{1}{2}\, x_2,\, \tfrac{1}{3}\, x_3,\, \dots).$$

Prove that $\text{ran}(T)$ is not topologically closed. (4p)