Applied Analysis (APPM 5440): Midterm 1
5.30pm – 6.45pm, Sep. 25, 2006. Closed books.

Problem 1: No motivation required for (a) and (c). Only brief motivations required for (b) and (d). 2 points each:

(a) Define what it means for a metric space (X, d) to be complete.

(b) Set $X = [0, 1] \cup [2, 3]$, and $\Omega = [0, 1]$. Is Ω open in the metric space $(X, |\cdot|)$?

(c) For $n \in \mathbb{N}$, set $x_n = e^{-1/n}\big(1 + (-1)^n\big) - 1/n$. Give numerical values for the quantities that exist among: $\lim_{n\to\infty} x_n$, $\limsup_{n\to\infty} x_n$, and $\liminf_{n\to\infty} x_n$.

(d) Construct a sequence $(x_n)_{n=1}^{\infty}$ such that $0 \leq x_n \leq 1$ for every n, and such that for any $\alpha \in [0, 1]$, there exists a subsequence $(x_{n_j})_{j=1}^{\infty}$ such that $x_{n_j} \to \alpha$ as $j \to \infty$.

(a) A metric space is complete if every Cauchy sequence in the space has a limit point in the space.

(b) Ω is open. To prove this, pick $x \in \Omega$, then $B_{1/2}(x) \subseteq \Omega$.[1]

(c) $\limsup x_n = 2$ and $\liminf x_n = 0$. $\lim x_n$ does not exist (since the limsup and the liminf are different).

(d) The set of all rational numbers in $[0, 1]$ is a countable set. Let $(x_n)_{n=1}^{\infty}$ denote an enumeration. This sequence satisfies the requirements.[2]

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[1] Note that

$$B_{1/2}(x) = \begin{cases} [0, x + 1/2) & \text{if } x < 1/2 \\ (0, 1) & \text{if } x = 1/2 \\ (x - 1/2, 1] & \text{if } x > 1/2. \end{cases}$$

In fact, Ω is both open and closed.

[2] The sequence

$$x_n = (0, 1/2,\quad 0, 1/4, 2/4, 3/4,\quad 0, 1/8, 2/8, 3/8, 4/8, 5/8, 6/8, 7/8,\quad 0, 1/16, \dots)$$

works as well.

Problem 2: Define a norm on $\mathbb{R}^d$ by setting, for $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$,

$$||x|| = \sum_{1 \le j \le d} |x_j|.$$

Using the fact that $(\mathbb{R}, |\cdot|)$ is complete, prove that $(\mathbb{R}^d, ||\cdot||)$ is complete. (3p)

Let $(x^{(n)})_{n=1}^{\infty}$ denote a Cauchy sequence in $\mathbb{R}^d$. We will prove that $(x^{(n)})$ has a limit point in $\mathbb{R}^d$.

First we construct the limit point x. For $j = 1, 2, \ldots, d$, we have

$$(1) \qquad |x_j^{(n)} - x_j^{(m)}| \le \sum_{j=1}^{d} |x_j^{(n)} - x_j^{(m)}| = ||x^{(n)} - x^{(m)}||.$$

Since $(x^{(n)})$ is a Cauchy sequence, it follows from (1) that $(x_j^{(n)})_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{R}$. Since $\mathbb{R}$ is complete, each such sequence has a limit point, name this point x_j. In other words,

$$(2) \qquad x_j = \lim_{n \to \infty} x_j^{(n)}.$$

Set $x = (x_1, x_2, \ldots, x_d)$. Clearly $x \in \mathbb{R}^d$.

Next we prove that the Cauchy sequence $(x^{(n)})$ converges to x. Fix an $\varepsilon > 0$. For each $j \in \{1, 2, \ldots, d\}$, equation (2) assures us that there exists an N_j such that

$$(3) \qquad n \ge N_j \quad \Rightarrow \quad |x_j^{(n)} - x_j| < \varepsilon/d.$$

Set $N = \max\{N_1, N_2, \ldots, N_d\}$. Then, if $n \ge N$, it follows from (3) that

$$||x^{(n)} - x|| = \sum_{j=1}^{d} |x_j^{(n)} - x_j| < \sum_{j=1}^{d} \varepsilon/d = \varepsilon.$$

Problem 3: Let (X, d_X), (Y, d_Y), and (Z, d_Z) denote metric spaces, and let $f: X \to Y$, and $g: Y \to Z$ denote continuous functions. Prove that the function $h: X \to Z$ that is defined by $h(x) = g(f(x))$ is continuous. (3p)

Let G denote an open set in Z. We will prove that h is continuous by proving that $h^{-1}(G)$ is necessarily open in X.

Since g is continuous, and G is open in Z, $g^{-1}(G)$ is open in Y.

Since f is continuous, and $g^{-1}(G)$ is open in Y, $f^{-1}(g^{-1}(G))$ is open in X.

Finally note that $h^{-1}(G) = f^{-1}(g^{-1}(G))$.

Problem 4: Let X denote the set of real numbers, and equip X with the discrete metric d_X (so that $d_X(x,y)=0$ if $x=y$, and $d_X(x,y)=1$ otherwise). Let (Y,d_Y) denote another metric space. For each statement below, either prove that it is necessarily true, or give a counter-example. (2p each.)

(a) Let f be a function from (X,d_X) to (Y,d_Y). Then f is necessarily continuous.

(b) Let g be a function from (Y,d_Y) to (X,d_X). Then g is necessarily continuous.

(a) Yes, f is necessarily continuous. To prove this, we fix an $x \in X$ and a number $\varepsilon > 0$. We will prove that there exists a $\delta > 0$ such that

$$d_X(x,y) < \delta \quad \Rightarrow \quad d_Y(f(x),f(y)) < \varepsilon.$$

Pick $\delta = 1/2$. Then if $d_X(x,y) < 1/2$, we must have $x = y$, and then of course $d_Y(f(x),f(y)) = 0 < \varepsilon$.

(b) No, f need not be continuous. As an example, set $Y = \mathbb{R}$ with the usual metric, and consider $g(x) = x$. Now if $x \in X$, then the set $\{x\}$ is open in (X,d_X)[3], but $g^{-1}(\{x\}) = \{x\}$ which is not open in (Y,d_Y).

[3] To see that $\{x\}$ is open, simply note that $B_{1/2}(x) = \{x\} \subseteq \{x\}$.

Problem 5: Let (X, d) denote a metric space, and let Y denote a subset of X. Consider the following three sets, and three statements:

Ω_1 is the set of all $x \in X$ for which there exists $(y_n)_{n=1}^{\infty} \subseteq Y$ such that $y_n \to x$.

$\Omega_2 = \bigcap_{\alpha \in A} F_\alpha$ where $\{F_\alpha\}_{\alpha \in A}$ is the set of all closed sets in (X, d) that contain Y.

$(\tilde{Y}, \tilde{d})$ is the completion of the metric space (Y, d).

(a) $\Omega_1 \subseteq \Omega_2$

(b) $\Omega_2 \subseteq \Omega_1$

(c) The two metric spaces (Ω_2, d) and $(\tilde{Y}, \tilde{d})$ are isometrically isomorphic.

For each statement, either prove that it is necessarily true, or give a counter-example (if you give a counter-example, you do not need to justify it in detail). You may not use any theorems given in class that relate to the concept of "closure". (2p each.)

(a) Assume that $x \in \Omega_1$. Then there exist points $(y_n)_{n=1}^{\infty} \subseteq Y$ such that $y_n \to x$. But then if F_α is a closed set that contains Y, it follows that $x \in F_\alpha$ since $(y_n) \subseteq F_\alpha$, and F_α contains all its limit points. Consequently, $x \in \Omega_2$.

(b) Assume that $x \in \Omega_2$. First we note that for every $\varepsilon > 0$, the set $B_\varepsilon(x) \cap Y$ is non-empty. (If there existed an $\varepsilon > 0$ such that $B_\varepsilon(x) \cap Y$ were empty, then $B_\varepsilon(x)^{\mathrm{c}}$ would be a closed set in the collection $(F_\alpha)_{\alpha \in A}$, and then x could not be a member of Ω_2.) Consequently, we can for $n = 1, 2, \dots$ pick $y_n \in B_{1/n}(x) \cap Y$. Then $y_n \to x$, and so $x \in \Omega_1$.

(c) This is not true. Consider the example $X = \mathbb{Q}$ with the usual metric, and $Y = \{q \in \mathbb{Q} : 0 \leq q \leq 1\}$. Then $\tilde{Y} = \{r \in \mathbb{R} : 0 \leq r \leq 1$ and $\tilde{d}$ is the usual metric on $\mathbb{R}$. Moreover, $\Omega_2 = Y$. The sets $(\tilde{Y}, \tilde{d})$ and (Ω_2, d) cannot be isometrically isomorphic since $\tilde{Y}$ is uncountable and Ω_2 is countable.

Note that if (X, d) is complete, then (Ω_2, d) is a completion of (Y, d), and since all completions are isometrically isomorphic, $(\tilde{Y}, \tilde{d})$ and (Ω_2, d) are isometrically isomorphic.

Here is an alternative proof for (a) and (b):

Pick an $x \in X$. Set $\varepsilon = \inf\{d(x, y) : y \in Y\}$. We will prove that if $\varepsilon > 0$, then x belongs to neither Ω_1 nor Ω_2; and if $\varepsilon = 0$, then x belongs to both Ω_1 and Ω_2. This proves that $\Omega_1 = \Omega_2$.

Case 1, $\varepsilon > 0$: No sequence in Y can converge to x, so $x \notin \Omega_1$. Moreover, $B_\varepsilon(x)^{\mathrm{c}}$ is a closed set that contains Y. Hence $x \notin \Omega_2$.

Case 2, $\varepsilon = 0$: In this case $B_\varepsilon(x) \cap Y$ is non-empty for every ε. By picking $y_n \in B_{1/n}(x) \cap Y$, we construct a sequence in Y such that $y_n \to x$. So $x \in \Omega_1$. This argument also shows that x belongs to any closed set F_α that contains Y, and consequently $x \in \Omega_2$.