Applied Analysis (APPM 5440): Midterm 1
5.30pm – 6.45pm, Sep. 25, 2006. Closed books.

Problem 1: No motivation required for (a) and (c). Only brief motivations required for (b) and (d). 2 points each:

(a) Define what it means for a metric space (X, d) to be complete.

(b) Set $X = [0,1] \cup [1,2]$, and $\Omega = [0,1]$. Is Ω open in the metric space $(X, |\cdot|)$?

(c) For $n \in \mathbb{N}$, set $x_n = e^{-1/n}\big(1 + (-1)^n\big) - 1/n$. Give numerical values for the quantities that exist among: $\lim_{n\to\infty} x_n$, $\limsup_{n\to\infty} x_n$, and $\liminf_{n\to\infty} x_n$.

(d) Construct a sequence $(x_n)_{n=1}^{\infty}$ such that $0 \leq x_n \leq 1$ for every n, and such that for any $\alpha \in [0,1]$, there exists a subsequence $(x_{n_j})_{j=1}^{\infty}$ such that $x_{n_j} \to \alpha$ as $j \to \infty$.

Problem 2: Define a norm on $\mathbb{R}^d$ by setting, for $x = (x_1, x_2, \dots, x_d) \in \mathbb{R}^d$,

$$||x|| = \sum_{1\leq j\leq d} |x_j|.$$

Using the fact that $(\mathbb{R}, |\cdot|)$ is complete, prove that $(\mathbb{R}^d, ||\cdot||)$ is complete. (3p)

Problem 3: Let (X, d_X), (Y, d_Y), and (Z, d_Z) denote metric spaces, and let $f : X \to Y$, and $g : Y \to Z$ denote continuous functions. Prove that the function $h : X \to Z$ that is defined by $h(x) = g(f(x))$ is continuous. (3p)

Problem 4: Let X denote the set of real numbers, and equip X with the discrete metric d_X (so that $d_X(x,y) = 0$ if $x = y$, and $d_X(x,y) = 1$ otherwise). Let (Y, d_Y) denote another metric space. For each statement below, either prove that it is necessarily true, or give a counter-example. (2p each.)

(a) Let f be a function from (X, d_X) to (Y, d_Y). Then f is necessarily continuous.

(b) Let g be a function from (Y, d_Y) to (X, d_X). Then g is necessarily continuous.

Problem 5: Let (X, d) denote a metric space, and let Y denote a subset of X. Consider the following three sets, and three statements:

Ω_1 is the set of all $x \in X$ for which there exists $(y_n)_{n=1}^{\infty} \subseteq Y$ such that $y_n \to x$.

$\Omega_2 = \bigcap_{\alpha\in A} F_\alpha$ where $\{F_\alpha\}_{\alpha\in A}$ is the set of all closed sets in (X, d) that contain Y.

$(\tilde{Y}, \tilde{d})$ is the completion of the metric space (Y, d).

(a) $\Omega_1 \subseteq \Omega_2$

(b) $\Omega_2 \subseteq \Omega_1$

(c) The two metric spaces (Ω_2, d) and $(\tilde{Y}, \tilde{d})$ are isometrically isomorphic.

For each statement, either prove that it is necessarily true, or give a counter-example (if you give a counter-example, you do not need to justify it in detail). You may not use any theorems given in class that relate to the concept of "closure". (2p each.)