Solutions to homework set 6 — APPM5440, Fall 2006

2.10: Let A denote the set of functions in $C(\mathbb{R}^n)$ that vanish at infinity. That $A = \overline{C_c}$ follows from the following two claims:

- Claim 1: C_c is dense in A.
- Claim 2: A is closed.

Proof of Claim 1: Fix an $f \in A$. We need to prove that for any $\varepsilon > 0$, there exists a $g \in C_c$ such that $||f-g||_u < \varepsilon$. Fix $\varepsilon > 0$. Set $R = \sup\{|x| : |f(x)| \geq \varepsilon\}$ (so that $|f(x)| \leq \varepsilon$ when $|x| \geq R$). Set for $x \in \mathbb{R}^n$

$$\varphi_R(x) = \begin{cases} 1 & |x| \in [0, R), \\ 1 + R - |x| & |x| \in [R, R+1], \\ 0 & |x| \in (R, \infty), \end{cases}$$

and set $g = f\,\varphi_R$. Then $g \in C_c$, and $||f - g||_u < \varepsilon$.

Proof of Claim 2: We will prove that $C(I)\backslash A$ is open. Fix an $f \in C(I)\backslash A$. Then for some $\varepsilon > 0$, there exist $(x_j)_{j=1}^{\infty} \in \mathbb{R}^n$ such that $|f(x_j)| \geq \varepsilon$ for all j, and $|x_j| \to \infty$. Then if $h \in C(I)$, and $||f - h|| < \varepsilon/2$, we find that

$$|h(x_j)| = |f(x_j) + (h(x_j) - f(x_j))| \geq |f(x_j)| - |h(x_j) - f(x_j)| > \varepsilon/2,$$

and so $h \in C(I)\backslash A$. It follows that $B_{\varepsilon/2}(f) \subseteq C(I)\backslash A$.

2.11: Set $g_n = f_n - f$. Then for every $x \in I$, $g_n(x) \searrow 0$. We need to prove that g_n converges uniformly to 0.

Since $g_n(x) \searrow 0$ for every x, $(||g_n||_u)_{n=1}^{\infty}$ is a decreasing sequence. Set $\alpha = \lim_{n\to\infty} ||g_n||_u$. If $\alpha = 0$, then $g_n \to 0$ uniformly. Assume $\alpha \neq 0$. Then for each $n = 1, 2, \dots$, there exists a point $x_n \in I$ such that $g_n(x_n) \geq \alpha$. Since I is compact, there exists an $x \in I$ and a subsequence n_j such that $x_{n_j} \to x$. Since $g_n(x) \searrow 0$, there exists an N such that $g_N(x) < \alpha/2$. Since g_N is continuous at x, there exists an $\varepsilon > 0$ such that $g_N(y) < 3\alpha/4$ for all $y \in B_\varepsilon(x)$. But then $g_n(y) < 3\alpha/4$ for all $n \geq N$ (since $g_n(y) \leq g_N(y)$ when $n \geq N$). This contradicts the claims that $g_{n_j}(x_{n_j}) \geq \alpha$, and $x_j \to x$ as $j \to \infty$.

2.12: Fix an $x \in [0, 1]$. Fix an $\varepsilon > 0$. Since $\Omega = \{f_n\}$ is equicontinuous, there exists a $\delta > 0$ such that if $|x - y| < \delta$, then $|f_n(x) - f_n(y)| < \varepsilon/2$. Now, if $|x - y| < \delta$, then

$$|f(x) - f(y)| = \lim_{n\to\infty} |f_n(x) - f_n(y)| \leq \limsup_{n\to\infty} \varepsilon/2 = \varepsilon/2.$$

2.14: Set

$$e(t) = |u(t) - u_0|.$$

Then e satisfies

$$e(t) = |u(t) - u(t_0)| = \left| \int_{t_0}^{t} f(s, u(s))\, ds \right| \leq \int_{t_0}^{t} |f(s, u(s))|\, ds. \tag{1}$$

Now use that

$$|f(s,u(s))| = |(f(s,u(s)) - f(s,u_0)) + f(s,u_0)| \tag{2}$$
$$\le |f(s,u(s)) - f(s,u_0)| + |f(s,u_0)|$$
$$\le K|u(s) - u_0| + M$$
$$= K\,e(s) + M.$$

Inserting (2) into (1) we find that

$$e(t) \le M|t-t_0| + \int_{t_0}^{t} K\,e(s)\,ds.$$

A direct application of Grönwall's inequality results in

$$e(t) \le M\,|t-t_0|\,e^{K\,|t-t_0|}.$$

For the last part of the problem, the exact solution of the given ODE is $u(t) = u_0\,e^{K(t-t_0)}$, and so

$$|u(t) - u_0| = |u_0|\,|e^{K(t-t_0)} - 1| \le |u_0|\,K|t-t_0|\,e^{K|t-t_0|},$$

since $|e^\alpha - 1| \le |\alpha| e^{|\alpha|}$ for all real α. Since in this example $f(t,u) = K\,u$, and $M = K\,|u_0|$, we see that the given solution satisfies the bound we proved.

Problem: Set $I = (0,\,1)$ and let $(f_n)_{n=1}^{\infty}$ be a sequence of continuously differentiable functions on I. Set $\Omega = \{f_n\,:\,1 \le n < \infty\}$.

(a) For a given n, suppose that

$$\sup_{x\in I} |f_n'(x)| < \infty.$$

Prove that then f_n is uniformly continuous.

(b) Suppose that

$$\sup_{x\in I}\ \sup_{1\le n<\infty} |f_n'(x)| < \infty.$$

Prove that then Ω is uniformly equicontinuous.

(c) Suppose that for every $x \in I$, there exists a $\kappa > 0$ such that

$$\sup_{1\le n<\infty}\ \sup_{y\in B_\kappa(x)} |f_n'(y)| < \infty.$$

Prove that then Ω is equicontinuous.

(d) Give an example of a set Ω of functions satisfying the condition in (c) that is not uniformly equicontinuous.

(e) Suppose that for a given $x \in I$, it is the case that

$$\sup_{1\le n<\infty} |f_n'(x)| < \infty.$$

Prove that Ω is not necessarily equicontinuous at x.

Solution: (a) Set $M_a = \sup_{x\in I} |f_n'(x)|$. Fix an $\varepsilon > 0$. Set $\delta = \varepsilon/M_a$. Then, if $|x - y| < \delta$, we have

$$|f_n(x) - f_n(y)| = |\int_x^y f_n'(y)\,dy| \leq \int_x^y M\,dy = M_a\,|x-y| < \varepsilon. \tag{3}$$

(b) Set $M_b = \sup_{x\in I} \sup_{1\leq n<\infty} |f_n'(x)|$. Fix a $\varepsilon > 0$. Set $\delta = \varepsilon/M_b$. Then if $f_n \in \Omega$, and $|x - y| < \delta$, an estimate analogous to (3) shows that $|f_n(x) - f_n(y)| < \varepsilon$.

(c) Fix an $x \in I$ and an $\varepsilon > 0$. We need to prove that there exists a $\delta > 0$ such that if $y \in B_\varepsilon(x)$, then $|f_n(x) - f_n(y)| < \varepsilon$ for any $f_n \in \Omega$. Set $M_c = \sup_{1\leq n<\infty} \sup_{y\in B_\kappa(x)} |f_n'(y)|$. Then $\delta = \min(\varepsilon/M_c, \kappa)$ works.

(d) $f_n(x) = n + 1/x$.

(e) Consider $f_n(x) = n\,(x-1/2)^2$. Then, at $x = 1/2$, we have $\sup_n f_n'(x) = 0$ but $\Omega = \{f_n\}$ is not equicontinuous. (Set $\varepsilon = 1$. Then given any δ, there exist n such that $f_n(x + \delta/2) > \varepsilon$; in fact any $n > 4\varepsilon/\delta^2$ will do.)