Homework set 6 — APPM5440

From the textbook: 2.10, 2.11, 2.12. If you have time, do 2.14.

Problem: Set $I = (0, 1)$ and let $(f_n)_{n=1}^{\infty}$ be a sequence of continuously differentiable functions on I. Set $\Omega = \{f_n : 1 \leq n < \infty\}$.

(a) For a given n, suppose that
$$\sup_{x \in I} |f_n'(x)| < \infty.$$
Prove that then f_n is uniformly continuous.

(b) Suppose that
$$\sup_{x \in I} \sup_{1 \leq n < \infty} |f_n'(x)| < \infty.$$
Prove that then Ω is uniformly equicontinuous.

(c) Suppose that for every $x \in I$, there exists a $\kappa > 0$ such that
$$\sup_{1 \leq n < \infty} \sup_{y \in B_\kappa(x)} |f_n'(y)| < \infty.$$
Prove that then Ω is equicontinuous.

(d) Give an example of a set Ω of functions satisfying the condition in (c) that is not uniformly equicontinuous.

(e) Suppose that for a given $x \in I$, it is the case that
$$\sup_{1 \leq n < \infty} |f_n'(x)| < \infty.$$
Prove that Ω is not necessarily equicontinuous at x.