Homework set 5 — APPM5440

2.4: Let's consider $X = [-1, 1]$ instead. Then set $f(x) = |x|$, and

$$f_n(x) = \frac{1 + n\,x^2}{\sqrt{n + n^2 x^2}}.$$

Then $f_n \to f$ uniformly, $f_n \in C^\infty(X)$, and f is not differentiable. (To justify the shift we made initially, simply note that if we define $g_n \in C([0,\,1])$ by $g_n(y) = f_n(2y-1)$, then g_n is an answer to the original problem.)

2.5: Set $I = [a, b]$. Let $(f_n)_{n=1}^\infty$ be a Cauchy sequence in $C^1(I)$. Since

$$||f_n - f_m||_{\rm u} \leq ||f_n - f_m||_{C^1},$$

the sequence (f_n) is Cauchy in $C(I)$. Since $C(I)$ is complete, there exists a function $f \in C(I)$ such that $f_n \to f$ uniformly.

Next set $g_n = f_n'$. Then

$$||g_n - g_m||_{\rm u} = ||f_n' - f_m'||_{\rm u} \leq ||f_n - f_m||_{C^1},$$

so (g_n) is Cauchy in $C(I)$. Therefore, there exists a function $g \in C(I)$ such that $g_n \to g$ uniformly.

It remains to prove that $f \in C^1(I)$, and that $f_n \to f$ in $C^1(I)$. Fix any $x \in I$, and any $h \in \mathbb{R}$ such that $x + h \in I$. Then

$$\begin{aligned}\frac{1}{h}\big(f(x+h) - f(x)\big) &= \lim_{n\to\infty} \frac{1}{h}\big(f_n(x+h) - f_n(x)\big)\\ &= \lim_{n\to\infty} \frac{1}{h}\int_0^h f_n'(x+t)\,dt\\ &= \lim_{n\to\infty} \frac{1}{h}\int_0^h g_n(x+t)\,dt.\end{aligned}$$

Now recall that uniform convergence on a finite interval implies convergence of integrals. Since $g_n \to g$ uniformly, we therefore find that

$$\frac{1}{h}\big(f(x+h) - f(x)\big) = \frac{1}{h}\int_0^h g(x+t)\,dt.$$

Since g is continuous, the limit as $h \to 0$ exists, and so

$$f'(x) = \lim_{h\to 0} \frac{1}{h}\big(f(x+h) - f(x)\big) = \lim_{h\to 0} \frac{1}{h}\int_0^h g(x+t)\,dt = g(x).$$

This proves that $f \in C^1(I)$. To prove that $f_n \to f$ in $C^1(I)$, we note that

$$||f - f_n||_{C^1} = ||f - f_n||_{\rm u} + ||f' - f_n'||_{\rm u} = ||f - f_n||_{\rm u} + ||g - g_n||_{\rm u}.$$

By the construction of f and g, it follows that $||f - f_n||_{C^1(I)} \to 0$.

2.7: Set $I = [0,\,1]$, and $\Omega = \{f \in C(I)\,:\, \mathrm{Lip}(f) \leq 1, \int f = 0\}$.

We will use the Arzelà-Ascoli theorem, of course.

The Lipschitz condition implies that Ω is equicontinuous. (To prove this, fix any $\varepsilon > 0$. Set $\delta = \varepsilon$. Then for any $f \in \Omega$, and $|x - y| < \delta$, we have $|f(x) - f(y)| \leq \text{Lip}(f)\,|x - y| \leq |x - y| < \varepsilon$.)

To prove that Ω is bounded, note that if $\int f = 0$, and f is continuous, then there must exist an $x_0 \in I$ such that $f(x_0) = 0$. Then for any $x \in I$ and any $f \in \Omega$, we have $|f(x)| = |f(x) - f(x_0)| \leq \text{Lip}(f)\,|x - x_0| \leq |x - x_0| \leq 1$. So $||f||_{\text{u}} \leq 1$.

Finally we need to prove that Ω is closed. Let (f_n) be a Cauchy sequence in Ω. Since $C(I)$ is complete, there exists an $f \in C(I)$ such that $f_n \to f$ uniformly. We need to prove that $f \in \Omega$. Since $f_n \to f$ uniformly, we know both that $\text{Lip}(f) \leq \limsup_{n\to\infty} \text{Lip}(f_n) \leq 1$, and that $\int f = \lim_{n\to\infty} \int f_n = 0$. This proves that $f \in \Omega$.

2.8: We will explicitly construct a dense countable subset Ω of $C([a,\, b])$. Without loss of generality, we can assume that $a = 0$ and that $b = 1$.

For $n = 1, 2, \dots$, and for $j = 0, 1, 2, \dots, n$, set $x_j^{(n)} = j/n$. Let Ω_n denote the subset of $C(I)$ of functions that (1) are linear on each interval $[x_{j-1}^{(n)}, x_j^{(n)}]$, and (2) take on rational values for each $x_j^{(n)}$. Since each function in Ω_n is uniquely defined by its values on the $x_j^{(n)}$'s, we can identify Ω_n by $\mathbb{Q}^n$. Hence Ω_n is countable.

Set $\Omega = \bigcup_{n=1}^{\infty} \Omega_n$. Since each Ω_n is countable, Ω is countable.

It remains to prove that Ω is dense in $C(I)$. Fix any $f \in C(I)$, and any $\varepsilon > 0$. Since I is compact, f is uniformly continuous on I so there exists a $\delta > 0$ such that $|x - y| < \delta$ implies that $|f(x) - f(y)| < \varepsilon/5$. Pick an n such that $1/n < \delta$, and pick a $\varphi \in \Omega_n$ such that $|\varphi(x_j^{(n)}) - f(x_j^{(n)})| < \varepsilon/5$ for $j = 0, 1, 2, \dots, n$. We will prove that $||\varphi - f||_{\text{u}} < \varepsilon$: Fix an $x \in I$. Then pick $j \in \{1, 2, \dots, n\}$ so that $x \in [x_{j-1}^{(n)},\, x_j^{(n)}]$. Then

$$|f(x) - \varphi(x)| \leq |f(x) - f(x_j^{(n)})| + |f(x_j^{(n)}) - \varphi(x_j^{(n)})| + |\varphi(x_j^{(n)}) - \varphi(x)|.$$

The first term is bounded by $\varepsilon/5$ due to the uniform continuity of f. The second term is bounded by $\varepsilon/5$ by the selection of φ. For the third term, we find that

$$\begin{aligned} |\varphi(x_j^{(n)}) - \varphi(x)| &\leq |\varphi(x_j^{(n)}) - \varphi(x_{j-1}^{(n)})| \\ &\leq |\varphi(x_j^{(n)}) - f(x_j^{(n)})| + |f(x_j^{(n)}) - f(x_{j-1}^{(n)})| + |f(x_{j-1}^{(n)}) - \varphi(x_{j-1}^{(n)})|. \end{aligned}$$

The first and the last terms are bounded by $\varepsilon/5$ by the selection of φ, and the middle term is bounded by $\varepsilon/5$ by the uniform continuity of f. It follows that $|f(x) - \varphi(x)| < \varepsilon$.

2.9: (a) Suppose that $w(x) > 0$ for $x \in (0,\, 1)$. We will verify that $||\cdot||_w$ is a norm:
(i) $||\lambda f||_w = \sup_x w(x)|\lambda f(x)| = |\lambda| \sup_x w(x)|f(x)| = |\lambda|\, ||f||_w$.
(ii) $||f + g||_w = \sup_x w(x)|f(x) + g(x)| \le \sup_x w(x)\big(|f(x)| + |g(x)|\big) \le \sup_x w(x)|f(x)| + \sup_x w(x)|g(x)| = ||f||_w + ||g||_w$.
(iii) If $f = 0$, then clearly $||f||_w = 0$. Conversely, if $f \neq 0$, then $f(x_0) \neq 0$ for some $x_0 \in (0,1)$. Then $||f||_w \ge w(x_0)|f(x_0)| > 0$.

(b) Assume that $w(x) > 0$ for $x \in [0,\, 1] =: I$. Set $m = \inf_{x\in I} w(x)$ and $M = \sup_{x\in I} w(x)$. Since I is compact and w is continuous, w attains both its inf and its sup, and therefore $m > 0$ and $M < \infty$. Then

$$||f||_{\mathrm{u}} = \sup_{x\in I} |f(x)| \ge \sup_{x\in I} \frac{w(x)}{M}|f(x)| = \frac{1}{M}||f||_w.$$

and

$$||f||_{\mathrm{u}} = \sup_{x\in I} |f(x)| \le \sup_{x\in I} \frac{w(x)}{m}|f(x)| = \frac{1}{m}||f||_w.$$

It follows that

$$\frac{1}{M}||f||_w \le ||f||_{\mathrm{u}} \le \frac{1}{m}||f||_w.$$

(c) Set $|||f||| = \sup_{x\in I} |x\, f(x)|$. We will prove that $|||\cdot|||$ is not equivalent to the uniform norm. Set for $n = 1, 2, \dots$

$$f_n(x) = \begin{cases} 1 & x \in [0, 1/n], \\ 0 & x \in (1/n, 1]. \end{cases}$$

Then

$$\inf_{||f||=1} |||f||| \le \inf_n |||f_n||| = \inf_n \frac{1}{n} = 0.$$

This proves that there cannot exist a $c > 0$ such that $|||f||| \ge c||f||$.

(d) We will prove that the set $C(I)$ equipped with the norm $|||\cdot|||$ is not a Banach space by constructing a Cauchy sequence with no limit point in $C(I)$. For $n = 1, 2, \dots,$ define $f_n \in C(I)$ by

$$f_n(x) = \begin{cases} x^{-1/2} & x \in (1/n, 1], \\ \sqrt{n} & x \in [0, 1/n]. \end{cases}$$

Fix a positive integer N. Then, if $m, n \ge N$, we have

$$\begin{aligned} |||f_n - f_m||| &= \sup_{x\in[0,1/N]} x\,|f_n(x) - f_m(x)| \\ &\le \sup_{x\in[0,1/N]} \big(x|f_n(x)| + x|f_m(x)|\big) \\ &\le \sup_{x\in[0,1/N]} \big(x/\sqrt{x} + x/\sqrt{x}\big) = 2N^{-1/2}. \end{aligned}$$

Consequently, $(f_n)_{n=1}^{\infty}$ is a Cauchy sequence. But f_n cannot converge uniformly to any function in $C(I)$. (To prove the last contention, suppose that

$f_n \to f$ for some $f \in C(I)$. Then $f(0) = \lim_{n\to\infty} f_n(0) = \infty$, which is a contradiction.)

Problem 1: Let $X = [0, \infty)$. Construct a sequence of functions $f_n : X \to \mathbb{R}$ that converges uniformly (and hence pointwise), but that does not converge in $L^2(X)$.

Solution: One possible choice is

$$\varphi_n(x) = \begin{cases} n^{-1/2} & x \in [0,\, n], \\ 0 & x \in (n, \infty). \end{cases}$$

Then $\varphi_n \to 0$ uniformly, but $||\varphi_n - 0|| = 1$ for all n.

Problem 2: Let $X = [0, 1]$. Construct a sequence of functions $f_n : X \to \mathbb{R}$ that converges in $L^2(X)$ but such that the sequence of numbers $(f_n(x))_{n=1}^{\infty}$ does not converge for *any* $x \in X$.

Solution: For $I = [a, b]$ an interval in X, consider the function

$$\chi_I(x) = \begin{cases} 1 & x \in I, \\ 0 & x \in X \backslash I. \end{cases}$$

Now construct intervals I_n that (1) decrease in size, and (2) march across the interval [0, 1]. For instance,

$$\begin{aligned} I_1 &= [0/2, 1/2], \\ I_2 &= [1/2, 2/2], \\ I_3 &= [0/4, 1/4], \\ I_4 &= [1/4, 2/4], \\ I_5 &= [2/4, 3/4], \\ I_6 &= [3/4, 4/4], \\ I_7 &= [0/8, 1/8], \\ I_8 &= [1/8, 2/8], \\ I_9 &= [2/8, 3/8], \\ &\vdots \end{aligned}$$

Set $\varphi_n = \chi_{I_n}$. Then $\varphi_n \to 0$ in L^2, but for any fixed x, the sequence of numbers $(\varphi_n(x))_{n=1}^{\infty}$ does not converge.