Homework set 3 solutions — APPM5440, Fall 2006

For problems 1.17, 1.18, 1.20, see attached notes.

Problem 1.22: First we prove that if $a, b \in S$, then either $C_a = C_b$, or the two sets are disjoint.

Case 1 - suppose that $a \sim b$. Then

$$c \in C_a \quad \Leftrightarrow \quad c \sim a \quad \Leftrightarrow \quad c \sim b \quad \Leftrightarrow \quad c \in C_b.$$

(In the middle equality, we used the assumption that $a \sim b$ and property (c).)

Case 2 - suppose that a and b are not equivalent. Then if $c \in C_a$, we know that $c \sim a$, and therefore c cannot be equivalent to b (since this would violate property (c)), and so $c \in C_b^{\mathrm{c}}$. Thus $C_a \subseteq C_c^{\mathrm{c}}$, which is to say that C_a and C_c are disjoint.

Next we prove that the relation

$$(x_n) \sim (y_n) \quad \Leftrightarrow \quad \lim d(x_n, y_n) = 0$$

is an equivalence relation. Properties (a) and (b) are obvious. To prove (c), assume that $(x_n) \sim (y_n)$ and that $(y_z) \sim (z_n)$. Then

$$\lim d(x_n, z_n) \leq \lim\big(d(x_n, y_n) + d(y_n, z_n)\big) = 0,$$

which proves that $(x_n) \sim (z_n)$.

Problem 1: Assume first that Ω is dense in X. Fix an $x \in X$. By our definition of denseness, we know that there exist $y_n \in \Omega$ such that $y_n \to x$. But then clearly there exist points y_n in any ε-ball around x.

Assume next that for any $x \in X$, and for any $\varepsilon > 0$, the set $\Omega \cap B_\varepsilon(x)$ is non-empty. We need to prove that Ω is dense in X. Fix any $x \in X$. For $n = 1, 2, 3, \dots$, pick $y_n \in B_{1/n}(x) \cap \Omega$. Then $y_n \to x$, and so $x \in \bar{\Omega}$. Since x was arbitrary, $X = \bar{\Omega}$.

Problem 2: Assume that (x_n) and (y_n) are Cauchy sequences in X. We will prove that $(d(x_n, y_n))$ is a Cauchy sequence in $\mathbb{R}$ (since $\mathbb{R}$ is complete, this implies that $d(x_n, y_n)$ converges). Fix $\varepsilon > 0$. Since both (x_n) and (y_n) are Cauchy, there exist N_1 and N_2 such that

$$m, n \geq N_1 \quad \Rightarrow \quad d(x_n, x_m) < \varepsilon/2, \tag{1}$$

and

$$m, n \geq N_2 \quad \Rightarrow \quad d(y_n, y_m) < \varepsilon/2. \tag{2}$$

Set $N = \max(N_1, N_2)$. Then, if $m, n \geq N$, we have

$$d(x_n, y_n) \leq d(x_n, x_m) + d(x_m, y_m) + d(y_m, y_n) < d(x_m, y_m) + \varepsilon, \tag{3}$$

and

$$d(x_m, y_m) \leq d(x_m, x_n) + d(x_n, y_n) + d(y_n, y_m) < d(x_n, y_n) + \varepsilon. \tag{4}$$

Together, (3) and (4) imply that $|d(x_m, y_m) - d(x_n, y_n)| < \varepsilon$.

Problem 3: Let $\tilde{x}, \tilde{y}, \tilde{z} \in \tilde{X}$. That $\tilde{d}(\tilde{x}, \tilde{y}) = \tilde{d}(\tilde{y}, \tilde{x})$ is obvious. Next assume that $\tilde{d}(\tilde{x}, \tilde{y}) = 0$, then if $(x_n) \in \tilde{x}$, and $(y_n) \in \tilde{y}$, we know that $\lim d(x_n, y_n) = 0$, which is to say that $(x_n) \sim (y_n)$ and so $\tilde{x} = \tilde{y}$. To finally prove the triangle inequality, pick representatives $(x_n) \in \tilde{x}$, $(y_n) \in \tilde{y}$, and $(z_n) \in \tilde{z}$. Then

$$\tilde{d}(\tilde{x}, \tilde{z}) = \lim d(x_n, z_n) \leq \lim\big(d(x_n, y_n) + d(y_n, z_n)\big) = \tilde{d}(\tilde{x}, \tilde{y}) + \tilde{d}(\tilde{y}, \tilde{z}).$$