

## Homework set 1 — APPM5440, Fall 2006

**Problem 2c:** Set  $I = [0, 1]$  and consider the set  $X$  consisting of all continuous functions on  $I$ , with the norm

$$\|f\| = \int_0^1 |f(x)| dx.$$

Prove that the space  $X$  is not complete.

**Solution:** A straight-forward way of proving this is to construct a Cauchy-sequence that does not have a limit point in  $X$ . One example is

$$f_n(x) = \begin{cases} -1 & x < 1/2 - 1/n, \\ n(x - 1/2) & 1/2 - 1/n \leq x \leq 1/2 + 1/n, \\ 1 & x > 1/2 + 1/n. \end{cases}$$

We first prove that  $(f_n)$  is Cauchy. Note that for any  $m, n$ , and  $x$ , we have  $|f_n(x) - f_m(x)| \leq 1$ . When  $m, n \geq N$ , we further have  $f_n(x) - f_m(x) = 0$  outside the interval  $[1/2 - 1/N, 1/2 + 1/N]$ , so

$$\|f_n - f_m\| = \int_{1/2-1/N}^{1/2+1/N} |f_n(x) - f_m(x)| dx \leq \int_{1/2-1/N}^{1/2+1/N} 1 dx = 2/N.$$

We next prove that  $(f_n)$  cannot converge to any element in  $X$ . Pick an arbitrary  $\varphi \in X$ . Assume temporarily that  $\varphi(1/2) \geq 0$ . Since  $\varphi$  is continuous, there exists a  $\delta > 0$  such that  $\varphi(x) \geq -1/2$  for  $x \in B_\delta(1/2)$ . Pick an integer  $N > 2/\delta$ . Then, for  $n \geq N$ , we have  $f_n(x) = -1$  when  $x \in [1/2 - \delta, 1/2 - \delta/2]$ , and so

$$\|f_n - \varphi\| \geq \int_{1/2-\delta}^{1/2-\delta/2} |f_n(x) - \varphi(x)| dx \geq \int_{1/2-\delta}^{1/2-\delta/2} 1/2 dx = \delta/4.$$

If on the other hand  $\varphi(1/2) < 0$ , then pick  $\delta > 0$  such that  $\varphi(x) \leq 1/2$  on  $[1/2, 1/2 + \delta]$  and proceed analogously.  $\square$

**Remark 1:** Note that you cannot solve a problem like the one above by constructing a Cauchy sequence  $(f_n)$  in  $X$ , point to a non-continuous function  $f$ , and claim that since  $f_n$  “converges to  $f$ ”,  $X$  cannot be complete. Note that the metric is *not even defined* for functions outside of  $X$ .

**Remark 2:** Can you somehow add the limit points of Cauchy sequences in  $X$  and obtain a complete space  $\tilde{X}$ ? The answer is yes, you can do that for any metric space; the resulting space  $\tilde{X}$  is called the “completion” of  $X$  and is (in a certain sense) unique. For the present example,  $\tilde{X}$  is the set of all (Lebesgue measurable) real-valued functions on  $I$  for which

$$\int_0^1 |f(x)| dx < \infty,$$

where the integral is what is called a “Lebesgue” integral. This space is denoted  $L^1(I)$ . Strictly speaking, an element of  $L^1(I)$  is an equivalence class of functions that differ only on a set of Lebesgue measure zero. This roughly means that two functions  $f$  and  $g$  are considered identical if

$$\int_0^1 |f(x) - g(x)| dx = 0.$$