Homework set 10 — APPM5440 — Solution sketches

Textbook 4.5a: The connected subsets of $\mathbb{R}$ are the intervals of the form (a,b), $[a,b]$, $(a,b]$, and $[a,b)$ where a and b are numbers such that $-\infty \leq a \leq b \leq \infty$. A full solution consists of two steps. First, let I denote and interval of the kind described and prove that I is connected (this is easily done via contradiction). Second, let Ω denote a subset that is not an interval, then you can construct two open disjoint subsets that cover Ω. This proves that Ω is not connected.

Textbook 4.6: Prove the following results:

- Let X and Y denote two homeomorphic topological spaces. Prove that X is connected if and only if Y is connected.
- Let X and Y denote two homeomorphic topological spaces, let $f:\ X \rightarrow Y$ denote a homeomorphism, and let $x \in X$. Prove that f is a homeomorphism between $X\backslash\{x\}$ and $Y\backslash\{f(x)\}$.
- Prove that $\mathbb{R}\backslash\{0\}$ is not connected.
- Prove that if $y \in \mathbb{R}^2$, then $\mathbb{R}^2\backslash\{y\}$ is connected.

Assume that $\mathbb{R}$ and $\mathbb{R}^2$ are connected. Derive a contradiction from the four facts given above.

Textbook 5.1: As an example, we prove that $a = b = c$, where

$$a = \sup_{x\neq 0} \frac{||Ax||}{||x||}, \qquad b = \sup_{||x||=1} ||Ax||, \qquad c = \sup_{x\leq 1} ||Ax||.$$

First we prove that $a = b$:

$$a = \sup_{x\neq 0} \frac{||Ax||}{||x||} = \sup_{x\neq 0} ||A\frac{x}{||x||}|| = \sup_{||y||=1} ||Ay|| = b.$$

It is obvious that $b \leq c$ (since the surface of the unit ball is a subset of the closed unit ball itself), so it only remains to prove that $c \leq b$. To this end, we pick a sequence of vectors x_n such that $||x_n|| \leq 1$ and $||Ax_n|| \rightarrow c$. Clearly, we can pick all x_n's to be non-zero. Then

$$c = \lim ||Ax_n|| \leq \limsup \frac{||Ax_n||}{||x_n||} = \limsup ||A\frac{x_n}{||x_n||}|| \leq \sup_{||y||=1} ||Ay|| = b.$$

Textbook 5.3: First note that

$$|\delta(f)| = |f(0)| \leq \sup_{x\in[0,1]} |f(x)| = ||f||_{\mathrm{u}}.$$

This immediately proves that δ is continuous w.r.t. the uniform norm.

A simple way to prove that δ is not continuous w.r.t. the L^1 norm is to construct a sequence of functions $f_n \in C([0,1])$ such that $||f_n||_{L^1} = 1$, but $|\delta(f_n)| = n$. For instance, the functions $f_n(x) = (n - n^2x/2)\,\chi_{[0,2/n]}(x)$ will do.

Problem 1: Set $X = \mathbb{R}^n$, $Y = \mathbb{R}^m$, and let $A \in \mathcal{B}(X, Y)$. Let

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

denote the representation of A in the standard basis. Equip X and Y with the supremum norms. Compute $||A||$.

Solution: This problem is solved in the text of the book.

Problem 2: Set $X = \mathbb{R}^2$ and $Y = \mathbb{R}$, and define $f : X \to Y$ by setting $f([x_1, x_2]) = x_1$. Prove that f is continuous. Prove that f is open. Prove that f does not necessarily map close sets to close sets.

Solution: First we prove that f is continuous. We use that in a metric space, continuity and sequential continuity are equivalent. Let $x^{(n)} \to x$ in $\mathbb{R}^2$, or, in other words, $(x_1^{(n)}, x_2^{(n)}) \to (x_1, x_2)$. Then it follows immediately that

$$f(x^{(n)}) = x_1^{(n)} \to x_1 = f(x).$$

Next we prove that f is open. Let $\Omega \subset \mathbb{R}^2$ be an open set. Pick a point x_1 in $f(\Omega)$. Then for some real number x_2, we have $x = (x_1, x_2) \in \Omega$. Since Ω is open, there exists $\varepsilon > 0$ such that $B_\varepsilon(x) \subseteq \Omega$. Then $(x_1 - \varepsilon, x_1 + \varepsilon) = f(B_\varepsilon(x)) \subseteq f(\Omega)$, and so $f(\Omega)$ must be open. (Draw a picture of all objects in this solution!)

Finally we prove that f is not closed via a counterexample. Consider $\Omega = \{(x_1, x_2) \in \mathbb{R}^2 : \, x_1 x_2 \geq 1\}$ (draw a picture!). Then Ω is closed in $\mathbb{R}^2$, but $f(\Omega) = (-\infty, 0) \cup (0, \infty)$ is not closed in $\mathbb{R}$.