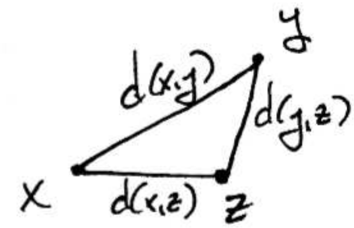
Metric Spaces

<u>Def^n</u> Let X be a non-empty set, and suppose that d is a function
$$d: X \times X \to [0, \infty)$$
such that



(i) $d(x,y) = 0 \iff x = y$
(ii) $d(x,y) = d(y,x) \quad \forall x,y \in X$
(iii) $d(x,y) \leq d(x,z) + d(z,y) \quad \forall x,y,z \in X$

Then (X, d) is called a METRIC SPACE with the METRIC d.

<u>Example</u> $\mathbb{R}^n$ with $d(x,y) = \|x-y\|$

<u>Example</u> $\mathbb{R}^n$ with $d(x,y) = \max_{1 \leq j \leq n} |x_j - y_j| \qquad x = (x_1, x_2, \ldots, x_n)$

<u>Proof</u> (i) & (ii) are obvious.
For (iii) we find
$$d(x,y) = \max_{1 \leq j \leq n} |x_j - y_j| = \max \cancel{\{} |(x_j - z_j) + (z_j - y_j)| \leq$$
$$\leq \max \left(|x_j - z_j| + |z_j - y_j| \right) \leq \max |x_j - z_j| + \max |z_j - y_j|$$
$$= d(x,z) + d(z,y)$$

<u>Example</u> Let X be any set.
Define $d(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y \end{cases}$

<u>Claim</u> Let $(\mathbb{X}, d)$ be a metric space.
Let $\mathbb{Y}$ be a subset of $\mathbb{X}$.
Then $(\mathbb{Y}, d)$ is a metric space.
We call $(\mathbb{Y}, d)$ a <u>subspace</u> of $(\mathbb{X}, d)$.

<u>Example</u> $\mathbb{X} = \mathbb{R}$ $\quad d(x,y) = |x-y|$
Then $(\mathbb{Q}, d)$ is a subspace of $(\mathbb{R}, d)$

<u>Def^n</u> Let F be a scalar field ($\mathbb{R}$ or $\mathbb{C}$).
Then a set $\mathbb{X}$ is called a VECTOR FIELD over F (or a LINEAR SPACE) if there exist operations "+" and "$\cdot$" such that

$\mathbb{X}$ is a commutative group w.r.t. +:

(i) $x+y = y+x \quad \forall x,y \in \mathbb{X}$
(ii) $(x+y)+z = x+(y+z) \quad \forall x,y,z \in \mathbb{X}$
(iii) $\exists\, 0 \in \mathbb{X}$ such that $x+0=x \quad \forall x \in \mathbb{X}$
(iv) $\forall x \in \mathbb{X}$ $\exists$ an element "$-x$" $\in \mathbb{X}$ s.t. $x+(-x)=0$

Conditions on scalar multiplication:

(v) $1 \cdot x = x \quad \forall x \in \mathbb{X}$
(vi) $(\lambda+\mu)x = \lambda x + \mu x \quad \forall \lambda, \mu \in F \quad x \in \mathbb{X}$
(vii) $\lambda(\mu x) = (\lambda\mu)x$ $\quad$ — " —
(viii) $\lambda(x+y) = \lambda x + \lambda y \quad \forall \lambda \in F \quad x,y \in \mathbb{X}$

Def^n A linear space $\mathfrak{X}$ (over F) is called a NORMED LINEAR SPACE if there exists a map $\|\cdot\| : \mathfrak{X} \to [0,\infty)$ such that

(i) $\|\lambda x\| = |\lambda| \, \|x\|$

(ii) $\|x+y\| \leq \|x\| + \|y\|$

(iii) $\|x\|=0 \iff x=0$

Claim A normed linear space is a metric space with the metric $d(x,y) = \|x-y\|$.

Example $\mathfrak{X} = \mathbb{R}^n$

$$\|x\|_p = \left(\sum_{j=1}^{d} |x_j|^p \right)^{1/p} \quad \text{for } p<\infty$$

$$\|x\|_\infty = \cancel{\sup} \max_{1 \leq j \leq n} |x_j| \quad \left(= \lim_{p \to \infty} \|x\|_p \right)$$

Claim $(\mathfrak{X}, \|\cdot\|_p)$ is a NLS when $1 \leq p \leq \infty$

Proof (i) & (iii) are trivial

(ii) is a little bit of work unless $p=1,2,\infty$

Homework!

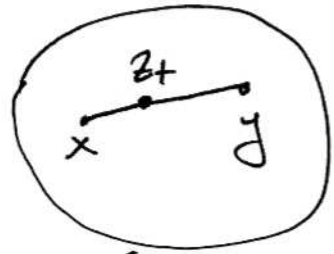
What about the case $p<1$? Let's take a detour:

Def^n Let A be a subset of a NLS $\mathfrak{X}$.

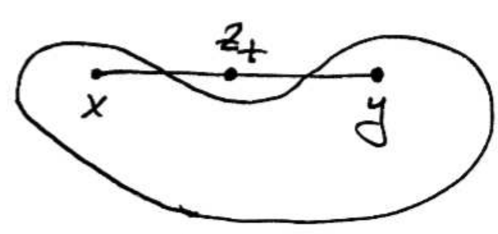
We say that A is CONVEX if $\forall x,y \in A$ & $t \in [0,1]$,

$z_t = tx + (1-t)y \in A$

Examples



Convex



Not convex

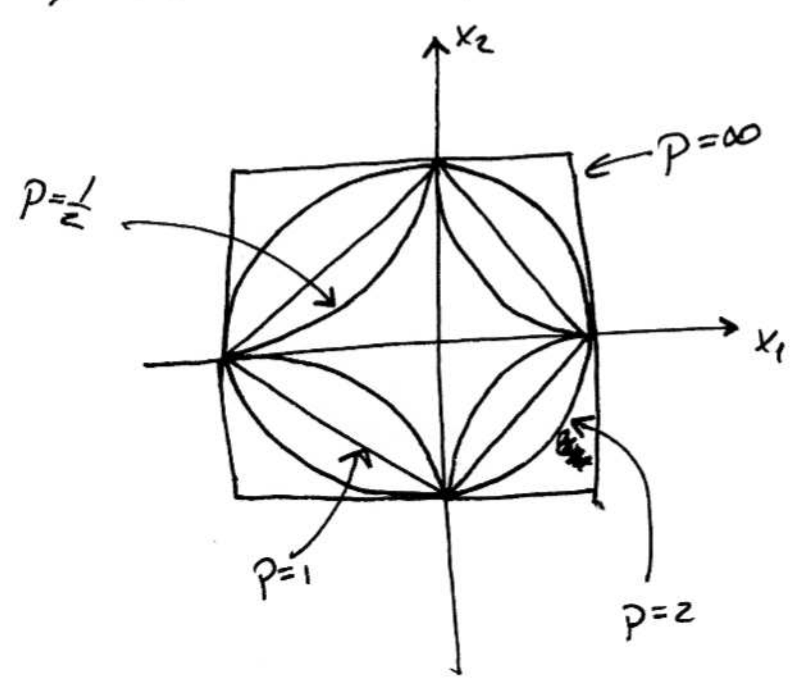
Lemma Let X be a NLS and set
$B = \{x \in X : \|x\| \le 1\}$ ← "Closed unit ball"
Then B is convex

Proof Pick $x, y \in B$ and $t \in [0,1]$. Then
$\|z_t\| = \|tx + (1-t)y\| \le \|tx\| + \|(1-t)y\| = t\|x\| + (1-t)\|y\| \le 1$
So $z_t \in B$.

Claim $(\mathbb{R}^n, \|\cdot\|_p)$ is NOT a NLS ~~was~~ when $p<1$.

Proof We will prove that the unit ball is not convex.
Set $e^{(1)} = [1,0,\dots,0]$, $e^{(2)} = [0,1,0,\dots,0]$, then $e^{(1)}$ & $e^{(2)} \in B$.
$z_{1/2} = [\frac{1}{2}, \frac{1}{2}, 0, 0, \dots, 0]$.
$\|z_{1/2}\|_p = \left(\frac{1}{2^p} + \frac{1}{2^p}\right)^{1/p} = \left(2\, 2^{-p}\right)^{1/p} = 2^{\frac{1}{p}-1} > 1$ if $p<1$
so $z_{1/2} \notin B$.

In $\mathbb{R}^2$, we find that the unit ball has the following shapes:



CONVERGENCE, COMPLETENESS

Def^n Let (X,d) be a metric space, and let $(x_n)_{n=1}^{\infty}$ be a sequence in X.

* We say that $x_n \to x$ if $d(x_n,x) \to 0$. CONVERGENCE
* We say that (x_n) is CAUCHY if $\forall \varepsilon > 0$ $\exists N$ such that $m,n \geq N \Rightarrow d(x_n,x_m) < \varepsilon$
* We say that X is COMPLETE if every Cauchy sequence in X has a limit point in X.

Note: Every convergent sequence is necessarily Cauchy.

Thm: $\mathbb{R}^n$ is a complete metric space. ← Important.

Example $X = (0,1)$. Is X complete? No: $x_n = \frac{1}{n}$ is Cauchy.

Example Every closed subset of $\mathbb{R}^n$ is a complete metric space. (We will return to the ~~d~~ question of when a set is "closed" shortly.)

Example $\mathbb{Q}$ with the usual metric is not complete. For a counterexample, pick for any integer n a rational ~~numb~~ number $q_n \in (\sqrt{2}-\frac{1}{n}, \sqrt{2}+\frac{1}{n})$. Then $(q_n)_{n=1}^{\infty}$ is Cauchy, but ~~not~~ does not converge to a point in $\mathbb{Q}$.

UPPER & LOWER BOUNDS

Def^n Let A be a subset of $\mathbb{R}$.

* If M is a number such that $x \le M \quad \forall x \in A$, then M is an UPPER BOUND of A.
* If M is an upper bound of A, and there is no smaller upper bound then we say that M is the *LEAST UPPER BOUND of A, $M = \sup A$
* If A does not have an upper bound, set $\sup A = \infty$
* Analogously, define LOWER BOUND, GREATEST LOWER BOUND, $\inf A$
* For the empty set, define $\sup \emptyset = -\infty$
* If $M = \sup A \in A$, set $M = \max A$

Thm Every subset of the real numbers has a unique supremum and infimum in the reals.

Note that this is not true for max/min.
Nor is it true for subsets of $\mathbb{Q}$, example: $\Omega = \{q \in \mathbb{Q} : q^2 < 2\}$

Def^n A sequence $(x_n)_{n=1}^{\infty} \subseteq \mathbb{R}$ is MONOTONE INCREASING if $x_n \le x_{n+1} \ \forall n$
—————— " —————— DECREASING if $x_n \ge x_{n+1} \ \forall n$

Every monotone sequence "converges" (possibly to $\pm\infty$),
if (x_n) is monotone increasing, then $\lim_{n\to\infty} x_n = \sup\{x_n\}_{n=1}^{\infty}$

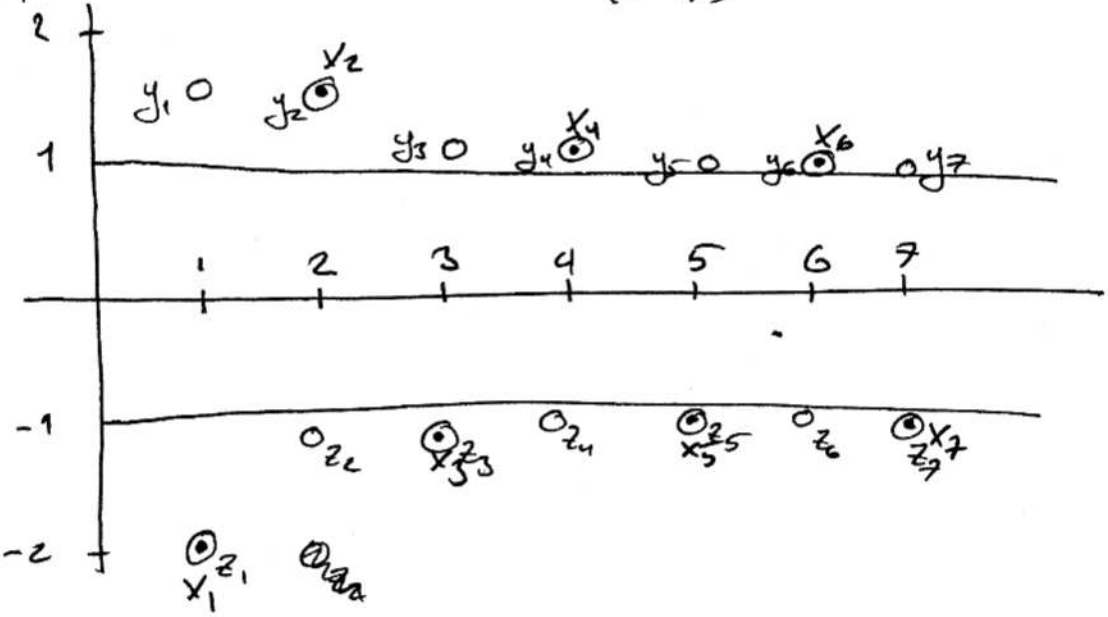
Not every sequence has a limit.
However, every sequence has what we call limsup & liminf.

Given a sequence $(x_n)_{n=1}^{\infty}$, set $y_n = \sup\{x_k : k \geq n\}$
$z_n = \inf\{x_k : k \geq n\}$

Then y_n is monotone decreasing $\Rightarrow$ $\lim y_n$ exists set $\lim y_n = \limsup$.

z_n is monotone increasing $\Rightarrow$ $\lim z_n$ exists. Set $\limsup x_n = \lim z_n$

Example: $x_n = (-1)^n \left(1 + \frac{1}{n}\right)$



$\limsup\limits_{n \to \infty} x_n = \lim y_n = 1$ $\qquad \liminf x_n = \lim z_n = -1$

Equivalent definitions:

$$\limsup_{n \to \infty} x_n = \lim_{n \to \infty} \sup\{x_k : k \geq n\}$$
$$= \inf\{\alpha : \text{Only finitely many } x_n \text{ are larger than } \alpha\}$$
$$= \sup\{\alpha : \text{there exists a subseq } (n_j)_{j=1}^{\infty} \text{ such that } \alpha = \lim_{j \to \infty} x_{n_j}\}$$

Lemma For <u>any</u> sequence $(x_n)_{n=1}^{\infty}$ in $\mathbb{R}$:

- $\limsup x_n$ and $\liminf x$ exist (but may equal $\pm\infty$)
- $\liminf x_n \leq \limsup x_n$
- (x_n) is convergent $\Leftrightarrow$ $\limsup x_n = \liminf x_n$
  In this case, $\lim x_n = \limsup x_n = \liminf x_n$, of course.

CONTINUITY

Def^n Let X and Y be metric spaces, and suppose $f: X \to Y$.

- We say that f is CONTINUOUS at x if $\forall \varepsilon > 0$, $\exists \delta > 0$ such that $d_X(x, x') < \delta \Rightarrow d_Y(f(x), f(x')) < \varepsilon$.
- We say that f is CONTINUOUS if it is cont. at every $x \in X$.
- We say that f is SEQUENTIALLY CONTINUOUS AT x if for every seq. (x_n) s.t. $x_n \to x$, we have $f(x_n) \to f(x)$
- We say that f is 'sequentially continuous' if f is seq cont. at every $x \in X$.

<u>Note</u>: We will soon prove that seq. cont $\Leftrightarrow$ cont.

<u>Example</u> Let X and Y be metric spaces.
Suppose that X has the discret metric, $d_X(x,y) = \begin{cases} 0 & \text{if } x = y \\ 1 & \text{if } x \neq y. \end{cases}$
Then <u>any</u> function $f: X \to Y$ is seq. cont

<u>Proof</u>: Suppose $x_n \to x$ in X. We need to prove that $f(x_n) \to f(x)$ in Y.
But if $x_n \to x$ in X, $\exists N$ s.t. $n \geq N \Rightarrow x_n = x$.
Then for $n \geq N$, $f(x_n) = f(x)$ so $f(x_n) \to f(x)$.

OPEN & CLOSED SETS

<u>Def^n</u> Let (X, d) be a metric space.

* For $c \in X$, $r \in [0,\infty)$, define $B_r(c) = \{x \in X : d(x,c) < r\}$ ← "open ball"
* ~~for $c \in X$, $r \in [0,\infty)$, define $\overline{B_r(c)} = \{x \in X : d(x,c) \leq$~~
* A set $G \subseteq X$ is OPEN if $\forall x \in G$ $\exists r > 0$ s.t. $B_r(x) \subseteq G$
* A set $F \subseteq X$ is CLOSED if $F^c = X \setminus F$ is open.
* The BOUNDARY of a set $\Omega \subseteq X$ is the set of all points $x \in X$ such that for any $\varepsilon > 0$, $B_\varepsilon(x)$ contains points in both Ω and Ω^c

<u>Prop^n</u> Let (X, d) be a metric space.

(i) X and $\emptyset$ are both open and closed.
(ii) A finite intersection of open sets is open.
(iii) Any union of open sets is open.
(iv) A finite union of closed sets is closed.
(v) Any intersection of closed sets is closed.

Note that the requirements of finiteness above is necessary.

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, 1\right) = [0,1)$$

$$\bigcap_{n=1}^{\infty} \left(-\frac{1}{n}, 1+\frac{1}{n}\right) = [0,1]$$

$$\bigcup_{n=1}^{\infty} \left[\frac{1}{n}, 1-\frac{1}{n}\right] = (0,1)$$

Proof of (iii) Let $\{G_\alpha\}_{\alpha\in A}$ be a collection of open sets and set $G=\bigcup_{\alpha\in A} G_\alpha$.
Fix an $x\in G$. $\exists\, \beta\in A$ s.t. $x\in G_\beta$.
Since G_β is open, $\exists\,\varepsilon>0$ s.t. $B_\varepsilon(x)\subseteq G_\beta$.
Since $G_\beta\subseteq G$, it follows that $B_\varepsilon(x)\subseteq G$.

Proof of (v): Let $\{F_\alpha\}_{\alpha\in A}$ be a collection of closed sets and set $F=\bigcap_{\alpha\in A} F_\alpha$.
That F is closed follows immediately from (iii) since
$F^c=\bigcup_{\alpha\in A} F_\alpha^c$ and all F_α^c are open.

Prop^n ~~Def^n~~

Let X & Y be metric spaces, and let f map X to Y.
TFAE: (a)* f is ε-δ-cont.
(b)* f is seq. cont.
(c) f is open-set cont.

Proof: We will prove that (a) $\Rightarrow$ (b) $\Rightarrow$ (c) $\Rightarrow$ (a)

(a)$\Rightarrow$(b) Suppose that f is ε-δ-cont.
We need to prove that if $x_n\to x$, then $f(x_n)\to f(x)$.
Suppose that $x_n\to x$. Fix $\varepsilon>0$.
f is ε-δ cont at x $\Rightarrow$ $\exists\,\delta$ s.t. $f(B_\delta(x))\subseteq B_\varepsilon(f(x))$.
Since $x_n\to x$, $\exists\, N$ s.t. $n\geq N \Rightarrow x_n\in B_\delta(x)$.
Then for $n\geq N$, $f(x_n)\in B_\varepsilon(f(x))$.

Note: The prop^n says that in a metric space, the three notions of continuity are equivalent. Henceforth, we will simply say "continuous".

(b)$\Rightarrow$(c) Suppose that f is NOT open-set cont.

We will construct a seq (x_n) s.t. $x_n \to x$ for some x, but $f(x_n) \not\to f(x)$.

Since f is not open-set cont, $\exists$ an open set $G \subseteq Y$ s.t. $H = f^{-1}(G)$ is not open.

Since H is not open, $\exists x \in H$ s.t. $B_\varepsilon(x) \cap H^c$ is non-empty for every $\varepsilon > 0$.

For $n = 1, 2, 3, \dots$ pick $x_n \in B_{1/n}(x) \cap H^c$.

Then $x_n \to x$.

However, since $f(x) \in G$, and G is open, $\exists \varepsilon$ s.t. $B_\varepsilon(f(x)) \in G$.

Since $f(x_n) \notin G$, it follows that $d(f(x_n), f(x)) > \varepsilon \ \forall n$ and so $f(x_n)$ cannot converge to $f(x)$.

(c)$\Rightarrow$(a) Suppose that f is open-set cont.

We will prove that f is ε-δ-cont. Fix an $x \in X$.

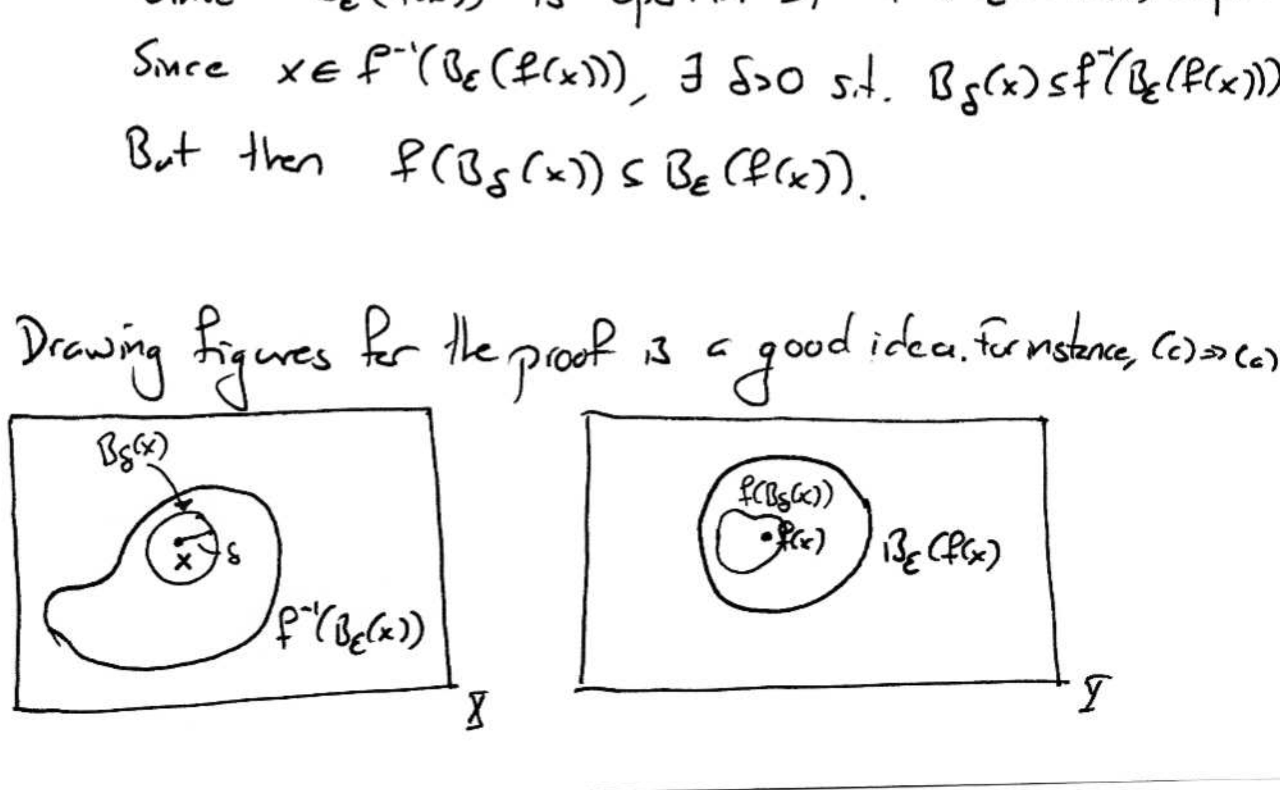
Pick any $\varepsilon > 0$.

Since $B_\varepsilon(f(x))$ is open in Y, $f^{-1}(B_\varepsilon(f(x)))$ is open in

Since $x \in f^{-1}(B_\varepsilon(f(x)))$, $\exists \delta > 0$ s.t. $B_\delta(x) \subseteq f^{-1}(B_\varepsilon(f(x)))$

But then $f(B_\delta(x)) \subseteq B_\varepsilon(f(x))$.

Note: Drawing figures for the proof is a good idea. For instance, (c)$\Rightarrow$(a)



Def^n Suppose that Ω is a subset of a metric space X.
The closure of Ω is the set $\overline{\Omega} = \{x \in X : \exists (x_n) \subset \Omega \text{ s.t. } x_n \to x\}$.

Claim Letting $\partial\Omega$ denote the boundary of Ω, we have $\overline{\Omega} = \Omega \cup \partial\Omega$.

Claim $\overline{\Omega}$ = the intersection of all closed sets containing Ω.

Claim Ω is closed $\iff \Omega = \overline{\Omega}$.

Example Let X denote the metric space $\mathbb{R}$ with the usual metric.
Let $\Omega = \mathbb{Q}$. Then $\overline{\Omega} = \mathbb{R} = X$.

Example Let X denote the set $\mathbb{R}$ and let $d(x,y) = \begin{cases} 0 & \text{if } x=y \\ 1 & \text{if } x \neq y. \end{cases}$
Let Ω be any subset of X. Then $\overline{\Omega} = \Omega$.
(Note that in the discrete metric, ANY set is closed.)

Def^n Let X be a metric space.
A subset Ω is said to be dense in X if $\overline{\Omega} = X$.

~~Example Under th~~

Def^n If a metric space has a dense countable subset, then we say that the metric space is seperable.

Example $\mathbb{R}$ equipped with the usual metric is seperable since $\mathbb{Q}$ is dense.

Example $\mathbb{R}$ equipped with the discrete metric is NOT seperable since no ~~sup~~ subset is dense (except $\mathbb{R}$ itself).

Example Let X denote the set of all sequences $x=(x_1, x_2, x_3, \dots)$ such that (i) $x_n \in \mathbb{Q}$ $\forall n$, and (ii) only finitely many x_n's are non-zero. Define for $x=(x_1, x_2, \dots)$ & $y=(y_1, y_2, y_3, \dots)$ the metric $d(x,y) = \left(\sum_{n=1}^{\infty} |x_n - y_n|^2 \right)^{1/2}$.

~~Then X is~~

Note that X is countable (to see this, note that $X = \bigcup_{n=1}^{\infty} X_n$, where X_n is the set of all sequences $(x_1, x_2, \dots, x_n, 0, 0, 0, \dots)$ where $x_j \in \mathbb{Q}$ $\forall j$, then each X_n is countable since it can be identified with $\mathbb{Q}^n$, and thus X is a countable union of countable sets).

Next ~~set~~ let $\tilde{X}$ denote the set of all sequences $x=(x_1, x_2, \dots)$ such that $x_n \in \mathbb{R}$ and $\sum_{n=1}^{\infty} |x_n|^2 < \infty$. Equip $\tilde{X}$ with the same metric as X.

~~Note that~~ X is dense in $\tilde{X}$. To prove this, fix any $x \in \tilde{X}$. $x=(x_1, x_2, \dots)$
Fix a j. set $\varepsilon = \frac{1}{j}$. Pick N_j s.t. $\sum_{n=N_j+1}^{\infty} |x_n|^2 < \frac{\varepsilon^2}{2}$.

Next, pick for $n=1,2,\dots,N_j$ numbers $x_n^{(j)} \in \mathbb{Q}$ such that $|x_n - x_n^{(j)}| < \frac{\varepsilon}{\sqrt{2N_j}}$, and set $x^{(j)} = (x_1^{(j)}, \dots, x_{N_j}^{(j)}, 0, 0, \dots)$.

Then $x^{(j)} \in X$, and $d(x, x^{(j)}) = \Big(\underbrace{\sum_{n=1}^{N_j} |x_n - x_n^{(j)}|^2}_{< \varepsilon^2/2} + \underbrace{\sum_{n=N_j+1}^{\infty} |x_n|^2}_{< \varepsilon^2/2} \Big)^{1/2} < \varepsilon = \frac{1}{j}$

Thus $x^{(j)} \to x$. ~~and so x~~

COMPLETION OF A METRIC SPACE

It sometimes happen that we define a set X and a metric d and find that the resulting space (X, d) is not complete. This is highly inconvenient.
It turns out to be possible to "add the missing elements" and obtain a new space $(\tilde{X}, \tilde{d})$ that is complete. This space is called the COMPLETION of (X, d).
- It is in a certain sense unique.

Example: The set of real numbers $\mathbb{R}$ can be defined by first defining the rational numbers $\mathbb{Q}$ and then form the completion of $\mathbb{Q}$ w.r.t. the metric $d(x, y) = |x - y|$.

Example: The spaces X & $\tilde{X}$ on page 13.

Example: The homework problem where $I = [0,1]$ and
X = the space of all continuous functions on I,
and $d(f, g) = \left(\int_I |f(x) - g(x)|^2 dx \right)^{1/2}$.
Then $\tilde{X} = L^2(I)$ = the space of all Lebesgue-measurable functions f s.t. $\int_0^1 |f(x)|^2 dx < \infty$.

Caveat: In $L^2(I)$, two functions f and g are considered identical if $\int |f(x) - g(x)|^2 dx = 0$.
To be precise, an element in $L^2(I)$ is an equivalence class of functions.

<u>Def^n</u> Let $\mathbb{X}$ and $\mathbb{Y}$ be metric spaces.
A map $i: \mathbb{X} \to \mathbb{Y}$ is called an <u>isometry</u> if
$$d_{\mathbb{X}}(x,y) = d_{\mathbb{Y}}(i(x), i(y)) \quad \forall x, y \in \mathbb{X}.$$
If is also ONTO, then i is ~~an~~ a metric space <u>isomorphism</u>.

<u>Note 1:</u> Any isometry is necessarily one-to-one
(~~Ex~~ If $i(x) = i(y)$, then $d(x,y) = d(i(x), i(y)) = 0 \Rightarrow x = y$.)

<u>Note 2:</u> If two spaces $\mathbb{X}$ and $\mathbb{Y}$ are isomorphic, i.e. if there exists an isomorphism from $\mathbb{X}$ to $\mathbb{Y}$, then mathematically speaking, $\mathbb{X}$ and $\mathbb{Y}$ are identical, they differ only in how the elements are labelled.
The concept of an isomorphism can be defined for metric space, NLS, inner-product spaces, Lie-algebras, etc, etc.
In each case, the isomorphism must preserve all the relevant structure.

<u>Example</u> $\mathbb{R}^2$ and $\mathbb{C}$ are isomorphic metricspaces $i: \mathbb{R}^2 \to \mathbb{C}: \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \mapsto x_1 + i x_2$

<u>Example</u> $\mathbb{X} = L^2(\underset{I=[-\pi,\pi]}{I}, \mathbb{R})$ = the space of "all" real-valued functions s.t. $\int |f|^2 < \infty$.
$\mathbb{Y} = l^2(\mathbb{Z}, \mathbb{C})$ = the space of all sequences $(\dots, \alpha_{-2}, \alpha_{-1}, \alpha_0, \alpha_1, \alpha_2, \dots)$ such that $\sum_{\infty} |\alpha_j|^2 < \infty$, $\alpha_j \in \mathbb{C}$.
$$d(\alpha, \beta) = \left(\sum_{j=-\infty}^{\infty} |\alpha_j - \beta_j|^2 \right)^{1/2}$$
The Fourier transform $F: \mathbb{X} \to \mathbb{Y}$ is ~~an isometry~~ defined by
$$F: f \mapsto (\dots, \alpha_{-1}, \alpha_0, \alpha_1, \dots) \text{ where } \alpha_n = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} e^{-inx} f(x)\,dx$$

Parseval's equality states that $\int |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |\alpha_n|^2$.

It follows that F is an isometry.

However f is not an isomorphism from $L^2(I, \mathbb{R})$ to $l^2(\mathbb{Z}, \mathbb{C})$ since, if $(\alpha_n) = Ff$, then $\overline{\alpha_n} = \alpha_{-n}$ (in other words, F is not <u>onto</u>).

However, F is an isomorphism from $L^2(I, \mathbb{C})$ - the set of all Lebesgue measurable square-integrable <u>complex-valued</u> functions to $l^2(\mathbb{Z}, \mathbb{C})$. Its invers is given by

$$[F^{-1}(\alpha_n)](x) = \sum_{n=-\infty}^{\infty} \alpha_n \frac{e^{inx}}{\sqrt{2\pi}}.$$

<u>Def</u>

<u>Def^n</u> A metric space $(\tilde{X}, \tilde{d})$ is a <u>completion</u> of the metric space (X, d) if

(a) There is an isometric embedding $i : X \to \tilde{X}$.

(b) $i(X)$ is dense in $\tilde{X}$.

(c) The space $(\tilde{X}, \tilde{d})$ is complete.

<u>Thm</u> Every metric space (X, d) has a completion $(\tilde{X}, \tilde{d})$. If $(\tilde{X}_1, \tilde{d}_1)$ and $(\tilde{X}_2, \tilde{d}_2)$ are both completions of (X, d), then $(\tilde{X}_1, \tilde{d}_1)$ and $(\tilde{X}_2, \tilde{d}_2)$ are isomorphic.

In order to prove the thm, we need the following lemma:

<u>Lemma</u> Let $(x_n)_{n=1}^{\infty}$ be a Cauchy seq. Then (x_n) has a subsequence $(y_j)_{j=1}^{\infty}$ s.t. $y_j = x_{n_j}$ and $m, n \geq N \Rightarrow d(y_n, y_m) \leq \frac{1}{N}$

<u>Proof</u> Let n_1 be s.t. $m, n \geq n_1 \Rightarrow d(x_m, x_n) \leq \frac{1}{1}$

Let n_2 be s.t. $m, n \geq n_2 \Rightarrow d(x_m, x_n) \leq \frac{1}{2}$ (and $n_2 \geq n_1$)

Let n_3 be s.t. $m, n \geq n_3 \Rightarrow d(x_m, x_n) \leq 1/3$ (and $n_3 \geq n_2$)

Etc

Then $y_j = x_{n_j}$ satisfies the criteria.

Proof of thm

Step 1: Prove uniqueness

Let $(\tilde{X}_1, \tilde{d}_1)$ & $(\tilde{X}_2, \tilde{d}_2)$ be completions, with isometries i_1 & i_2.
First we define an isometry $\tilde{i} : \tilde{X}_1 \to \tilde{X}_2$

$$\begin{tikzcd}
 & (\tilde{X}_1, \tilde{d}_1) \arrow[d, "\tilde{i}"] \\
(X, d) \arrow[ur, "i_1"] \arrow[r, "i_2"'] & (\tilde{X}_2, \tilde{d}_2)
\end{tikzcd}$$

For $\tilde{x} \in \tilde{X}_1$, pick $x_n \in X$ s.t. $i_1(x_n) \to \tilde{x}$ (possible since $i_1(X)$ is dense in $\tilde{X}_1$)

$(i_1(x_n))$ is Cauchy $\underset{\substack{\uparrow \\ i_1 \text{ is an isometry}}}{\Rightarrow}$ (x_n) is Cauchy $\underset{\substack{\uparrow \\ i_2 \text{ is an isometry}}}{\Rightarrow}$ $(i_2(x_n))$ is Cauchy

Since $\tilde{X}_2$ is complete $\exists \tilde{y} \in \tilde{X}_2$ s.t. $i_2(x_n) \to \tilde{y}$.
The element $\tilde{y}$ does not depend on the choice of sequence (x_n);
in fact, if $i_1(z_n) \to \tilde{x}$, then $\tilde{d}_2(i_2(z_n), i_2(x_n)) = \tilde{d}_1(i_1(z_n), i_1(x_n)) \to 0$
so $i_2(z) \to \tilde{y}$ as well. Therefore, we can define $\tilde{i}(\tilde{x}) = \tilde{y}$.

* We need to prove that $\tilde{i}$ is an isometry:
Pick $\tilde{x}, \tilde{y} \in \tilde{X}_1$ and $x_n, y_n \in X$ s.t. $i_1(x_n) \to \tilde{x}$ & $i_1(y_n) \to \tilde{y}$. Then
$\tilde{d}_1(\tilde{x}, \tilde{y}) = \lim \tilde{d}_1(i_1(x_n), i_1(y_n)) = \lim d(x_n, y_n) = \lim \tilde{d}_2(i_2(x_n), i_2(y_n)) = \tilde{d}_2(\tilde{i}(\tilde{x}), \tilde{i}(\tilde{y}))$.

* We need to prove that $\tilde{i}(i_1(X)) = i_2(X)$.
Simply use constant sequences!

* We need to prove that $\tilde{i}$ is ONTO.
Fix any $\hat{x} \in \tilde{X}_2$. Since $i_2(X)$ is dense in $\tilde{X}_2$, $\exists x_n \in X$
s.t. $i_2(x_n) \to \hat{x}$. Then $(i_2(x_n))$ is Cauchy in $\tilde{X}_2 \Rightarrow$
$\Rightarrow (x_n)$ is Cauchy in $X \Rightarrow (i_1(x_n))$ is Cauchy in $\tilde{X}_1$.
Since $\tilde{X}_1$ is complete, $\exists \tilde{x} \in \tilde{X}_1$ s.t. $i_1(x_n) \to \tilde{x}$.
But then $\tilde{i}(\tilde{x}) = \hat{x}$ by def^n.

Step 2: Construct the completion

We define $\tilde{X}$ as a set of equivalence classes on the set of all Cauchy sequences on X.

Suppose that (x_n) & (y_n) are Cauchy seq^s in X.

We say that $(x_n) \sim (y_n)$ if $\lim_{n\to\infty} d(x_n, y_n) = 0$ (this limit always exists → HOMEWORK)

This equivalence relation defines a set of equivalence classes which we label $\tilde{X}$. Given an equivalence class ~~$\tilde{x}$~~ $\tilde{x} \in \tilde{X}$, we say that a seq $(x_n) \in \tilde{x}$ is a <u>representative</u> of $\tilde{x}$.

* <u>Def^n of $\tilde{d}$</u>: Given $\tilde{x}, \tilde{y} \in \tilde{X}$, pick $(x_n) \in \tilde{x}$ & $(y_n) \in \tilde{y}$ and set $\tilde{d}(\tilde{x}, \tilde{y}) = \lim_{n\to\infty} d(x_n, y_n)$

We need to prove that $\tilde{d}(\tilde{x}, \tilde{y})$ does not depend on the choice of representative.

Suppose $(x_n') \in \tilde{x}$ & $(y_n') \in \tilde{y}$.

Then $d(x_n, y_n) \le \underbrace{d(x_n, x_n')}_{\to 0} + d(x_n', y_n') + \underbrace{d(y_n', y_n)}_{\to 0}$

so $\lim_{n\to\infty} d(x_n, y_n) \le \lim_{n\to\infty} d(x_n', y_n')$

Analogously, $\lim d(x_n', y_n') \le \lim d(x_n, y_n)$.

The proof that $\tilde{d}$ is a metric is left as a homework.

* <u>Construct the embedding $i: X \to \tilde{X}$.</u> Given an $x \in X$, set for $n = 1, 2, 3, \ldots$ $x_n = x$. Then (x_n) is a Cauchy seq.

Define $i(x)$ as the equivalence class containing this seq.

Step 3: Prove that $i(X)$ is dense in $\tilde{X}$

Fix any $\tilde{x} \in \tilde{X}$.

Pick a representative $(x_n) \in \tilde{x}$.

For $n = 1, 2, 3, \ldots$ let $\tilde{x}^{(n)}$ denote the equivalence class that contains the constant sequence $(x_n, x_n, x_n, \ldots)$.

Then clearly, $\tilde{x}^{(n)} \in i(X)$.

Moreover, $\lim_{n\to\infty} \tilde{d}(\tilde{x}^{(n)}, \tilde{x}) = \lim_{n\to\infty} \lim_{j\to\infty} d(x_j^{(n)}, x_j) = \lim_{n\to\infty} \lim_{j\to\infty} d(x_n, x_j) = 0$

so $\tilde{x}^{(n)} \to \tilde{x}$ in $\tilde{X}$.

Step 4 - Prove that $\tilde{X}$ is complete

Let $(\tilde{x}^{(n)})$ be a Cauchy seq in $\tilde{X}$.

Let $(\tilde{y}^{(n)})$ be a subseq of $(\tilde{x}^{(n)})$ s.t. $\tilde{d}(\tilde{y}^{(n)}, \tilde{y}^{(m)}) \leq \frac{1}{N}$ when $m, n \geq N$

For each $n = 1, 2, 3, \ldots$ pick a representative $(y_k^{(n)})_{k=1}^{\infty} \in \tilde{y}^{(n)}$.

We can pick the representative so that $d(y_k^{(n)}, y_l^{(n)}) \leq \frac{1}{N}$ when $k, l \geq N$.

Then $(y_k^{(k)})_{k=1}^{\infty}$ is a Cauchy seq in X. ~~since~~

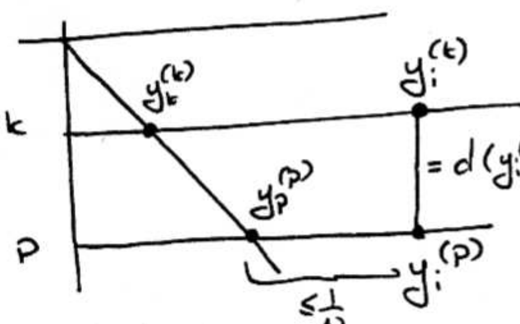
To prove this we first note that for any $i = 1, 2, \ldots$

$$d(y_k^{(k)}, y_p^{(p)}) \leq d(y_k^{(k)}, y_i^{(k)}) + d(y_i^{(k)}, y_i^{(p)}) + d(y_i^{(p)}, y_p^{(p)})$$

Suppose that $k, p \geq N$ and take the limit as $i \to \infty$:

$$d(y_k^{(k)}, y_p^{(p)}) \leq \underbrace{\lim_{i\to\infty} d(y_k^{(k)}, y_i^{(k)})}_{\leq \frac{1}{N}} + \underbrace{\lim_{i\to\infty} d(y_i^{(k)}, y_i^{(l)})}_{= \underbrace{\tilde{d}(\tilde{y}^{(k)}, \tilde{y}^{(l)})}_{\leq \frac{1}{N}}} + \underbrace{\lim_{i\to\infty} d(y_i^{(p)}, y_p^{(p)})}_{\leq \frac{1}{N}} \leq \frac{3}{N}$$

so $k, p \geq N \Rightarrow d(y_k^{(k)}, y_p^{(p)}) \leq \frac{3}{N}$ which means that $(y_k^{(k)})$ is Cauchy.



$= d(y_i^{(k)}, y_i^{(p)}) \xrightarrow[i\to\infty]{} \tilde{d}(\tilde{y}^{(k)}, \tilde{y}^{(p)}) \leq \frac{1}{N}$

Let $\tilde{x}$ denote the equivalence class containing $(y_k^{(k)})_{k=1}^{\infty}$. Then

$$\tilde{d}(\tilde{y}^{(n)}, \tilde{x}) = \lim_{k\to\infty} d(y_k^{(n)}, y_k^{(k)}) \leq \lim_{k\to\infty}\lim_{i\to\infty}\Big(\underbrace{d(y_k^{(n)}, y_i^{(n)})}_{\leq \frac{1}{\min(k,i)}} + d(y_i^{(n)}, y_i^{(k)}) + \underbrace{d(y_i^{(k)}, y_k^{(k)})}_{\leq \frac{1}{\min(k,i)}}\Big) \leq$$

$$\leq \lim_{k\to\infty}\Big(\frac{1}{k} + \underbrace{\tilde{d}(\tilde{y}^{(n)}, \tilde{y}^{(k)})}_{\leq \frac{1}{\min(n,k)}} + \frac{1}{k}\Big) = \frac{1}{n} \to 0 \text{ as } n \to \infty.$$

This proves that $\tilde{y}^{(n)} \to \tilde{x}$ in $\tilde{X}$; and since $\tilde{y}^{(n)}$ is a subseq. of the Cauchy seq. $(\tilde{x}^{(n)})$, it follows that $\tilde{x}^{(n)} \to \tilde{x}$.

COMPACTNESS

Compactness is a property that a subset of a metric space may possess.
(In $\mathbb{R}^n$ a subset is compact iff it is both closed and bounded.)

If K is a compact subset of a metric space, then

* Any cont function from K to a metric space $\mathcal{Y}$ is uniformly cont.
* Any sequence $(x_n)_{n=1}^{\infty}$ in K has a convergent subseq.
* Any Cauchy seq $(x_n)_{n=1}^{\infty}$ in K has a limit point in K.
* Any continuous function $f: K \to \mathbb{R}$ attains its max and min values.
* ~~Etc~~ etc...

Compactness is a very desirable property for a set to possess.

Now to the formal definition:

Def^n Let $(\mathcal{X}, d)$ be a metric space, and let $\Omega \subseteq \mathcal{X}$.

* We say that a collection of open sets $\{G_\alpha\}_{\alpha \in A}$ is an open cover of Ω if $\Omega \subseteq \bigcup_{\alpha \in A} G_\alpha$.
* We say that Ω is compact is every open cover has a finite subcover (i.e. if it is possible to pick a finite number of the open sets that cover Ω by themselves).

For a set Ω to be compact, it must be "totally bounded"

Def^n Let $(\mathcal{X}, d)$ be a metric space and let $\Omega \subseteq \mathcal{X}$. We say that Ω is totally bounded if for every $\varepsilon > 0$, Ω has an open cover of ε-balls.

Note 1: Any set that is ~~bdd is also total~~ totally bdd is also bdd.

Note 2: In $\mathbb{R}^n$, bounded $\Leftrightarrow$ totally bounded. (We will prove this shortly.)

Thm: Let $(\mathcal{X}, d)$ be a metric space, and let $K \subseteq \mathcal{X}$. TFAE:

(1) K is complete and totally bounded.
(2) Every sequence in K has a subseq that converges to an element in K.
(3) Every open cover of K has a finite subcover (i.e. K is compact).

Proof We will prove that
$(1) \Rightarrow (2)$
$(2) \Rightarrow (1)$
$(1) \& (2) \Rightarrow (3)$
$(3) \Rightarrow (2)$

* $(1) \Rightarrow (2)$ Suppose that (1) holds and that (x_n) is a seq. in K.
Since (1) holds, there exists a finite open cover of K consisting of balls of radius 1. At least one of those balls contain infinitely many points in the sequence (x_n). Denote this ball B_1, and the points inside B_1 $(x_n^{(1)})_{n=1}^{\infty}$.

Next, note that $B_1 \cap K$ can be covered by finitely many balls of radius 1/2. At least one ball contains infinitely many points in $(x_n^{(1)})_{n=1}^{\infty}$. Denote this ball B_2 and the points $(x_n^{(2)})_{n=1}^{\infty}$.

Repeat ... construct sequences $(x_n^{(j)})_{n=1}^{\infty}$

Set $y_n = x_n^{(n)}$. Then $(y_n)_{n=1}^{\infty}$ is Cauchy, and a subseq of (x_n)
Since K is complete, $\exists\, x \in K$ s.t. $y_n \to x$.

$(2) \Rightarrow (1)$ Assume that (1) is false. Then either one of two options pertain:

Case I: K is not complete. Then $\exists$ a Cauchy seq $(x_n) \subset K$ with no limit point in K. But then clearly no subsequence may converge.

Case II: K is not totally bdd. Then $\exists \varepsilon > 0$ s.t. no finite collection of ε-balls may cover K.
Pick x_1 arbitrarily.
Pick $x_2 \in K \setminus B_\varepsilon(x_1)$.
Pick $x_3 \in K \setminus (B_\varepsilon(x_1) \cup B_\varepsilon(x_2))$
Pick $x_n \in K \setminus (\bigcup_{j=1}^{n} B_\varepsilon(x_j))$.
Then $d(x_n, x_m) > \varepsilon$ when $n \neq m$ so no subsequence of $(x_n)_{n=1}^{\infty}$ can possibly converge.

(1) & (2) $\Rightarrow$ (3) Assume that (1) & (2) hold and that $(G_\alpha)_{\alpha\in A}$ is an open cover.

To be proved. $\rightarrow$ Claim: For some $\varepsilon>0$, it is the case that every ε-ball is wholly contained in a single G_α.

For the ε given in the claim, pick a finite cover $(B_\varepsilon(x_j))_{j=1}^N$ (this is possible since K is totally bounded.)

For each j, pick α_j s.t. $B_\varepsilon(x_j)\subseteq G_{\alpha_j}$.

Then $\{G_{\alpha_j}\}_{j=1}^N$ is a cover since $K\subseteq \bigcup_{j=1}^N B_\varepsilon(x_j)\subseteq \bigcup_{j=1}^N G_{\alpha_j}$

Proof of claim: Suppose the claim is not true. Then for each $n=1,2,\dots$ there exists a ball B_n of radius less than $1/n$ such that $B_n\cap K$ is non-empty and B_n is not contained in any single G_α. For each n, pick $x_n\in B_n\cap K$. By (2), $\exists$ a subseq $(x_{n_j})_{j=1}^\infty$ such that $x_{n_j}\to x$ as $j\to\infty$ for some $x\in K$. But then $x\in G_\beta$ for some β, so $\exists\varepsilon$ s.t. $B_\varepsilon(x)\subset G_\beta$. Now pick N such that $N>\frac{1}{\varepsilon/3}$ and for $n\geq N$, $d(x,x_n)<\frac{\varepsilon}{3}$.

Then $B_N\subseteq B_\varepsilon(x)\subseteq G_\alpha$ which is a contradiction.

(3) $\Rightarrow$ (2) We ~~pro~~ assume that (2) is false and prove that then (3) is false.

If (2) is false, then $\exists$ a seq $(x_n)\subseteq K$ with no convergent subsequence.

For each $x\in K$, $\exists\ \varepsilon_x>0$ s.t. $B_{\varepsilon_x}(x)$ contains at most finitely many points in (x_n).

$\{B_{\varepsilon_x}(x)\}_{x\in K}$ is an open cover of K.

Since any finite subcover contains only finitely many points in (x_n), no such cover may cover K.

Lemma: A subset of $\mathbb{R}^n$ is totally bounded if and only if it is bounded.

Proof That a set is bounded if it is totally bounded is obvious.
~~Next~~ We need to prove that if a set $\Omega \subseteq \mathbb{R}^n$ is bounded, then it is totally bounded. Let $\Omega \subseteq \mathbb{R}^n$ be a bounded set, and fix $\varepsilon > 0$.
$\exists\ R \in (0, \infty)$ s.t. $\Omega \subseteq [-R, R]^n$.
Let p denote a positive integer (to be determined) and divide $[-R, R]^n$ into p^n equisized cubes.
Each such cube has sidelength $\frac{2R}{p}$, and fits inside a closed ball of radius $\frac{\sqrt{n}R}{p}$.
Thus, if we pick $p > \frac{\sqrt{n}R}{\varepsilon}$, then each cube fits inside an open ball of radius ε. We have now constructed an open cover of $[-R, R]^n$, and hence of Ω, consisting of p^n balls of radius ε.

Recall that a subset of a complete metric space is complete itself if and only if it is closed. Since $\mathbb{R}^n$ is complete, we find that if Ω is a subset of $\mathbb{R}^n$, then

$$\Omega \text{ is closed} \Leftrightarrow \Omega \text{ is complete}$$
$$\Omega \text{ is bounded} \Leftrightarrow \Omega \text{ is totally bounded.}$$

Since we proved earlier that in a general metric space

$$\Omega \text{ is compact} \Leftrightarrow \Omega \text{ is complete and totally bounded.}$$

We obtain the classical result:

Prop^n Let Ω be a subset of $\mathbb{R}^n$. Then

$$\Omega \text{ is compact} \Leftrightarrow \Omega \text{ is closed and bounded.}$$

Def^n Let X be a metric space and let $\Omega \subseteq X$. Then we say that Ω is pre-compact if $\overline{\Omega}$ is compact.

Claim If X is a complete metric space, and $\Omega \subseteq X$, then
Ω is pre-compact $\Leftrightarrow$ Ω is totally bounded.

Thm Let X and Y be metric spaces, let $K \subseteq X$ be compact, and let $f: K \to Y$ be a continuous map. Then $f(K)$ is compact in Y.

Proof: Let $\{G_\alpha\}_{\alpha \in A}$ be an open cover of $f(K)$.
Since f is continuous, $\{f^{-1}(G_\alpha)\}_{\alpha \in A}$ is an open cover of K.
Since K is compact, we can find a finite subcover $\{f^{-1}(G_{\alpha_j})\}_{j=1}^{J}$ of K.
But then $\{G_{\alpha_j}\}_{j=1}^{J}$ is a finite cover of $f(K)$.

Corollary: Let X, Y, K, f be as above. Then f is bounded.

If $Y = \mathbb{R}$, then it is also simple to prove that f attains its max and min:

Thm Let X be a metric space, ~~and~~ let K be a compact subset of X, and let $f: K \to \mathbb{R}$ be a continuous map.
Then $\exists$ points $x_{max}, x_{min} \in K$ such that

$$f(x_{max}) = \sup_{x \in K} f(x) \qquad f(x_{min}) = \inf_{x \in K} f(x)$$

Proof We will construct the point x_{max}, the proof for x_{min} is analogous.
Pick a sequence $(x_n)_{n=1}^{\infty} \subset K$ such that $\lim_{n \to \infty} f(x_n) = \sup_{x \in K} f(x)$.
Since K is compact, (x_n) has a convergent subseq (x_{n_j}) with a limit point in K. Let this limit point be x_{max}.
Then $x_{n_j} \to x_{max}$ as $j \to \infty$, and since f is continuous,

$$f(x_{max}) = \lim_{j \to \infty} f(x_{n_j}) = \sup_{x \in K} f(x).$$

Thm Let X & Y be metric spaces, let $K \subseteq X$ be a compact set, and let $f: X \to Y$ be a continuous function.
Then f is uniformly continuous.

(Recall the def^n of uniform continuity: $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $d(x,y) < \delta \Rightarrow d(f(x), f(y)) < \varepsilon$)

Proof Assume that f is not uniformly continuous.
This means that for some $\varepsilon > 0$, there exist for every $\delta > 0$ points $x, y \in K$ s.t. $d(x,y) < \delta$ but $d(f(x), f(y)) > \varepsilon$.
For $n = 1, 2, 3, \dots$ pick $x_n, y_n \in K$ s.t. $d(x_n, y_n) < \frac{1}{n}$ and $\underbrace{d(f(x_n), f(y_n)) > \varepsilon}_{(*)}$.
Since K is compact, $\exists x \in K$, and a subseq (x_{n_j}) of (x_n) s.t. $x_{n_j} \to x$ as $j \to \infty$.
Then $d(y_{n_j}, x) \le d(y_{n_j}, x_{n_j}) + d(x_{n_j}, x) \le \frac{1}{n_j} + d(x_{n_j}, x) \to 0$ as $j \to \infty$
so $y_{n_j} \to x$ as $j \to \infty$.
If f were continuous, then we would have $\begin{Bmatrix} \lim_{j\to\infty} f(x_{n_j}) = f(x) \\ \lim_{j\to\infty} f(y_{n_j}) = f(x) \end{Bmatrix}$
but this is impossible in view of $(*)$.
Thus, we have proven that if f is not uniformly cont, then it is not cont.